Steiner Tree Problem

Given: Graph $G = (V, E)$, terminals $U \subseteq V$, edge costs $c_e \geq 0$, $e \in E$.

Task: Find subtree of G of minimum cost that contains all terminals in U.

LP relaxation:

$$\begin{aligned}
\min \quad & \sum_{e \in E} c_e \cdot x_e \\
\text{s.t.} \quad & \sum_{e \in \delta(S)} x_e \geq 1 && \text{for all } S \subseteq V \text{ with } \emptyset \neq S \cap U \neq U, \\
& x_e \geq 0 && \text{for all } e \in E.
\end{aligned}$$

Later, we will prove the following theorem.

Theorem 4.7.

There is a polynomial-time algorithm which computes a solution of value at most $2 \cdot \mathrm{OPT}_{LP}$.

Prize-Collecting Steiner Tree Problem

Given: Graph $G = (V, E)$, root node $r \in V$, edge costs $c_e \geq 0$, $e \in E$, and penalties $\pi_i \geq 0$, $i \in V$.

Task: Find subtree T containing root r minimizing $\sum_{e \in E(T)} c_e + \sum_{i \in V \setminus V(T)} \pi_i$.

Remark: The Steiner Tree Problem is a special case with $\pi_i = 0$ for all non-terminals and $\pi_i = \infty$ for terminals i.

IP formulation:

$$\begin{aligned}
\min \quad & \sum_{e \in E} c_e \cdot x_e + \sum_{i \in V} \pi_i \cdot (1 - y_i) \\
\text{s.t.} \quad & \sum_{e \in \delta(S)} x_e \geq \max_{i \in S} y_i && \text{for all } S \subseteq V \setminus \{r\}, \\
& y_r = 1, \\
& x_e, y_i \in \{0, 1\} && \text{for all } e \in E,\ i \in V.
\end{aligned}$$

LP relaxation: $x_e \geq 0$ for all $e \in E$ and $y_i \leq 1$ for all $i \in V$.

Deterministic LP Rounding Algorithm

Let $0 \leq \alpha < 1$.

1. compute optimal LP solution (x^*, y^*);
2. set $U := \{i \in V \mid y_i^* \geq \alpha\}$;
3. find Steiner tree T on terminals U using algorithm from Theorem 4.7;

Lemma 4.8.

The tree T returned by the algorithm has cost at most $\frac{2}{\alpha} \sum_{e \in E} c_e \cdot x_e^*$.

Proof:. . . □

Theorem 4.9.

For $\alpha = 2/3$ the cost of the solution returned by the algorithm is

$$\sum_{e \in E(T)} c_e + \sum_{i \in V \setminus V(T)} \pi_i \leq \frac{2}{\alpha} \sum_{e \in E} c_e \cdot x_e^* + \frac{1}{1-\alpha} \sum_{i \in V} \pi_i \cdot (1 - y_i^*) \leq 3 \cdot \text{OPT} \; .$$

Proof:. . . □