Existence of an FPTAS

We state the next theorem without proof:

Theorem 3.7.
There is a fully polynomial-time approximation scheme for the problem of minimizing the makespan on constantly many identical parallel machines.

Theorem 3.8.
If the number of machines is part of the input, there is no FPTAS, unless $P = NP$.

Proof:... □

Remark: More generally, a strongly NP-hard optimization problem whose objective function values are integral and polynomially bounded in the numbers occurring in the input does not have an FPTAS, unless $P = NP$.

Bin-Packing Problem

Given: n items with positive sizes $a_1, \dots, a_n \leq 1$.

Task: Pack the items into a minimal number of unit-size bins.

Theorem 3.9.
Unless $P = NP$, there is no α-approximation algorithm for the Bin-Packing Problem for any $\alpha < 3/2$.

Proof: Reduce the Partition Problem. □

Algorithm Next-Fit

- consider items in arbitrary order; start to pack them into the first bin;
- whenever next item does not fit into the current bin, open a new bin;

Theorem 3.10.
Algorithm Next-Fit runs in $O(n)$ time and uses at most $2 \cdot \mathrm{OPT} - 1$ bins.

Proof:... □

First-Fit Heuristics for Bin-Packing

Algorithm First-Fit

- consider items in arbitrary order; open one bin;
- pack the next item into the first open bin in which it fits;
- if the item does not fit into any open bin, open a new bin;

Theorem 3.11.

Algorithm First-Fit runs in polynomial time; it uses at most $\left\lceil \frac{17}{10}\mathrm{OPT}\right\rceil$ bins.

Proof: See, e. g., book of Korte & Vygen. □

Algorithm First-Fit-Decreasing

- consider items in order of decreasing size; open one bin;
- pack the next item into the first open bin in which it fits;
- if the item does not fit into any open bin, open a new bin;

Theorem 3.12.

Algorithm First-Fit-Decreasing uses at most $\frac{11}{9}\mathrm{OPT} + 4$ bins. □

Towards an Asymptotic PTAS for Bin-Packing

Definition 3.13.

An asymptotic polynomial-time approximation scheme (APTAS) is a family of polynomial-time algorithms $(A_\varepsilon)_{\varepsilon>0}$ along with a constant c such that A_ε returns a solution of value at most $(1+\varepsilon)\mathrm{OPT} + c$.

Lemma 3.14.

Any packing of all items of size $\geq \gamma$ into ℓ bins can be greedily extended to a packing of all items into at most $\max\left\{\ell, \frac{1}{1-\gamma}\mathrm{SIZE}(I) + 1\right\}$ bins.

Proof:. . . □

Remarks:

- For $\gamma = \varepsilon/2$, the lemma yields a packing of all items into at most $\max\left\{\ell, (1+\varepsilon)\mathrm{OPT} + 1\right\}$ bins.
- In the following we can thus restrict to items of size at least $\varepsilon/2$.

Linear Grouping Scheme

For given instance I and parameter $k \in \mathbb{Z}_{>0}$, define a new instance I':

1. sort the items of a given instance I such that $a_1 \geq a_2 \geq \cdots \geq a_{n-k}$;
2. instance I' has $n-k$ items of size $a'_i := a_{\lceil i/k \rceil \cdot k+1}$, $i = 1, \ldots, n-k$;

Remarks

- Instance I' has at most $\lfloor n/k \rfloor$ distinct item sizes.
- It holds that $a_{i+k} \leq a'_i \leq a_i$ for $i = 1, \ldots, n-k$.

Lemma 3.15.

Any packing of I' can be easily turned into a packing of I with at most k additional bins. Moreover,

$$\mathrm{OPT}(I') \leq \mathrm{OPT}(I) \leq \mathrm{OPT}(I') + k \ .$$

Proof:. . . □

APTAS for Bin-Packing

Ingredients:

- All items have size at least $\varepsilon/2$ such that $\mathrm{SIZE}(I) \geq \varepsilon n/2$.
- W.l.o.g.: $\varepsilon \cdot \mathrm{SIZE}(I) \geq 1$ (otherwise, there are at most $2/\varepsilon^2$ items).
- Set $k := \lfloor \varepsilon \cdot \mathrm{SIZE}(I) \rfloor$ and apply the linear grouping scheme.
- Resulting instance I' has at most $n/k \leq 4/\varepsilon^2$ distinct item sizes.
- Thus, instance I' can be solved optimally in polynomial time.

Theorem 3.16.

For any $\varepsilon > 0$, there is a polynomial-time algorithm for the Bin-Packing Problem that computes a solution with at most $(1+\varepsilon) \cdot \mathrm{OPT} + 1$ bins.

Proof:. . . □