Primal-Dual Algorithm

Note: The two previous algorithms require solving a linear program. Special purpose algorithms are often much faster!

Idea: Construct a feasible dual solution that is "good enough".

Primal-dual algorithm for the Set Cover Problem:

1. set $y :\equiv 0$ and $I := \emptyset$;
2. while $\exists e_k \notin \bigcup_{j \in I} S_j$
3.   increase y_k until $\exists j$ with $e_k \in S_j$ such that $\sum_{i : e_i \in S_j} y_i = w_j$;
4.   set $I := I \cup \{j\}$;

Theorem 1.7.

The primal-dual algorithm is an f-approximation algorithm for the Set Cover Problem.

Proof: As before. □

Greedy Algorithm

Idea: Iteratively select a set minimizing the ratio of its weight to the number of currently uncovered elements it contains.

Greedy algorithm for the Set Cover Problem

1. set $I := \emptyset$ and $\hat{S}_j := S_j$ for all j;
2. while I is not a cover
3. $\ell := \text{argmin}\left\{ \frac{w_j}{|\hat{S}_j|} \mid \hat{S}_j \neq \emptyset \right\}$;
4.   set $I := I \cup \{\ell\}$;
5.   set $\hat{S}_j := \hat{S}_j \setminus S_\ell$ for all j;

Theorem 1.8.

The greedy algorithm returns a cover I with $w(I) \leq H_g \cdot z^*_{LP}$, where $g := \max_j |S_j|$ and $H_g := \sum_{k=1}^{g} \frac{1}{k} \approx \ln\ g$.

Proof: Use technique called dual fitting... □

Nonapproximability Results for Set Cover

Theorem 1.9 (Lund & Yannakakis 1994,).

If there is a $c \ln n$-approximation algorithm for the Unweighted Set Cover Problem for some constant $c < 1$, then there is an $O(n^{O(\log \log n)})$-time deterministic algorithm for each NP-complete problem.

Theorem 1.10 (Feige 1998).

There is some constant $c > 0$ such that if there is a $c \ln n$-approximation algorithm for the Unweighted Set Cover Problem, then $P = NP$.

Theorem 1.11 (Dinur & Safra 2002).

If there is an α-approximation algorithm for the Vertex Cover Problem with $\alpha < 10\sqrt{5} - 21 \approx 1.36$, then $P = NP$.

Theorem 1.12 (Khot & Regev 2008).

Assuming the Unique Games Conjecture holds, if there is an α-approximation algorithm for the Vertex Cover Problem with $\alpha < 2$, then $P = NP$.

Chapter 2: Greedy Algorithms and Local Search

(cp. Williamson & Shmoys, Chapter 2)

Scheduling Jobs with Deadlines on a Single Machine

Given: n jobs $j = 1, \ldots, n$ with processing time $p_j \geq 0$, release date $r_j \geq 0$ and due dates d_j, $j = 1, \ldots, n$.

Task: Schedule each job nonpreemptively for p_j units of time, starting no earlier than time r_j, such that no two jobs overlap.

Objective: Minimize the maximum lateness $L_{\max} := \max_{j=1,\ldots,n} L_j$ with $L_j := C_j - d_j$ where C_j denotes the completion time of job j, $j = 1, \ldots, n$.

Theorem 2.1.

Deciding whether $L^*_{\max} \leq 0$ is strongly NP-complete.

Proof: Polynomial transformation of the 3-Partition Problem. □

Corollary 2.2.

There is no α-approximation algorithm for the scheduling problem for any α, unless $P = NP$.

Proof:. . . □