Chapter 9: The Hardness of Approximation

(cp. Williamson & Shmoys, Chapter 16)

MaxClique Problem

Given: Undirected graph $G = (V, E)$.

Task: Find $V' \subseteq V$ maximizing $|V'|$ with all nodes in V' pairwise adjacent.

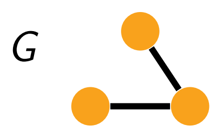
Notation: The size of a largest clique $V' \subseteq V$ in G is denoted by $\omega(G)$.

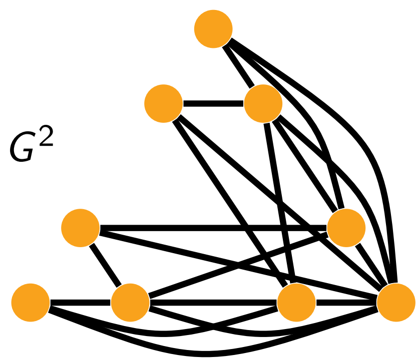
Definition 9.1 (product graph).

For an undirected graph $G = (V, E)$ let $G^k = (V^k, E_k)$ where V^k is the set of all k-tuples of nodes in V and E_k is defined by

$$E_k := \{(u_1, \dots, u_k)(v_1, \dots, v_k) \mid u_i = v_i \text{ or } u_i v_i \in E \text{ for all } i\} \ .$$

Example:





A Constant-Factor Approximation for MaxClique?

Lemma 9.2.

$$\omega(G^k) = \omega(G)^k$$

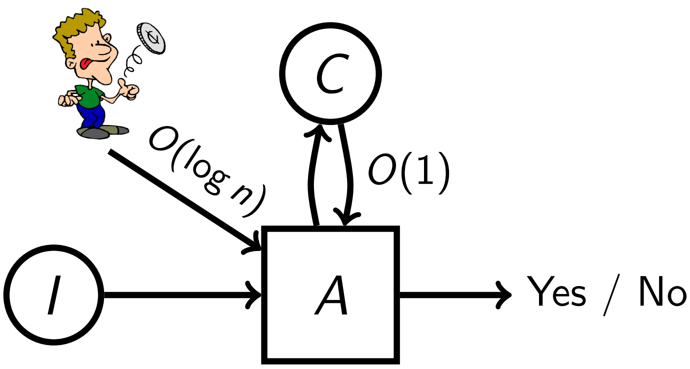
Proof:. . . □

Observation 9.3.

If there is an α-approximation algorithm for MaxClique for some fixed $\alpha < 1$, then there is a polynomial-time approximation scheme.

Proof:. . . □

Another Characterization of NP



correct answer is "Yes" $\implies$ $\exists$ certificate C: Pr(A outputs "Yes")$= 1$.

correct answer is "No" $\implies$ $\forall$ certificates C: Pr(A outputs "Yes")$< \frac{1}{2}$.

PCP Theorem (Arora, Lund, Motwani, Sudan, and Szegedy 1992)

Every decision problem in NP has a probabilistically checkable proof of constant query complexity and logarithmic randomness complexity.

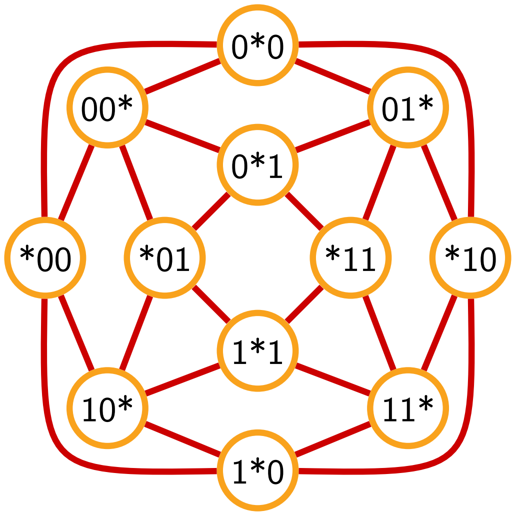
Hardness of Approximation

Theorem 9.4 (Feige, Goldwasser, Lovász, Safra, Szegedy 1991).

There is no 2-approximation algorithm for MaxClique, unless P=NP.

Proof: Form graph G representing PCP system for SAT instance such that

- $\omega(G) = K$ if correct answer is "Yes"
- $\omega(G) < \frac{K}{2}$ if correct answer is "No" □



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Theorem 9.5 (Zuckerman 2007).

There is no $n^{1-\varepsilon}$-approximation algorithm for MaxClique, for any $\varepsilon > 0$, unless $P = NP$.

Bottomline: MaxClique is one of the hardest problems to approximate!

Approximability of MAX 2SAT

In Chapter 5 we stated the following hardness result:

Theorem 9.6.

Unless $P = NP$, there is no $(7/8 + \varepsilon)$-approximation algorithm for MAX E 3SAT for any constant $\varepsilon > 0$.

Proof: Via PCP Theorem (see Williamson & Shmoys, Chapter 16.3). □

Theorem 9.7.

Unless $P = NP$, there is no α-approximation algorithm for MAX 2SAT for $\alpha > \frac{433}{440} \approx 0.984$.

Proof:. . . □

Approximation-Preserving Reductions

Consider two optimization problems $\mathcal{P}$ and $\mathcal{P}'$ with corresponding sets of instances $X_{\mathcal{P}}$ and $X_{\mathcal{P}'}$, respectively.

Definition 9.8 (L-Reduction).

An L-reduction from $\mathcal{P}$ to $\mathcal{P}'$ with parameters $a, b > 0$ is a map $f : X_{\mathcal{P}} \to X_{\mathcal{P}'}$ such that for all $I \in X_{\mathcal{P}}$:

i. $I' := f(I)$ can be computed in time polynomial in the size of I;
ii. $\mathrm{OPT}(I') \leq a \cdot \mathrm{OPT}(I)$;
iii. given a solution of value V' to I', one can compute in polynomial time a solution of value V to I such that
$$|\mathrm{OPT}(I) - V| \leq b \cdot |\mathrm{OPT}(I') - V'| \ .$$

Example: The reduction from MAX E 3SAT to MAX 2SAT in the proof of Theorem 9.7 is an L-reduction with parameters $a = \frac{55}{7}$ and $b = 1$.

Approximation-Preserving Reductions

Theorem 9.9.

For maximization problems $\mathcal{P}$ and $\mathcal{P}'$, if there is an L-reduction from $\mathcal{P}$ to $\mathcal{P}'$, and there is an α-approximation algorithm for $\mathcal{P}'$, then there is an $\big(1 - ab(1-\alpha)\big)$-approximation algorithm for $\mathcal{P}$.

Proof:… □

Theorem 9.10.

For minimization problems $\mathcal{P}$ and $\mathcal{P}'$, if there is an L-reduction from $\mathcal{P}$ to $\mathcal{P}'$, and there is an α-approximation algorithm for $\mathcal{P}'$, then there is an $\big(ab(\alpha-1)+1\big)$-approximation algorithm for $\mathcal{P}$. □

Examples of L-Reductions

Lemma 9.11.

There is an L-reduction with parameters $a = \frac{27}{7}$ and $b = 1$ from MAX E3SAT to MAX 2SAT.

Proof:… □

Corollary 9.12.

Unless $P = NP$, there is no α-approximation algorithm for MAX 2SAT for $\alpha > \frac{209}{216} \approx 0.968$.

Lemma 9.13.

There is an L-reduction with parameters $a = 2\Delta$ and $b = 1$ from Vertex Cover in bounded degree graphs to the Steiner Tree Problem.

Proof:… □