Algorithmic Game Theory — Extra Assignment Sheet

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Due: February 10, 2016

Exercise 7.1. Let $G = (\{1,2\}, \mathbf{S}, \mathbf{u})$ be a finite two-player game. Consider the following (non-standard) mixed extensions $\tilde{G} = (\{1,2\}, \tilde{\mathbf{S}}, \tilde{\mathbf{u}})$ where $\tilde{S}_i = \Delta(S_i)$ for $i = 1, 2$. Each mixed strategy $\mathbf{x} \in \square(\mathbf{S})$ imposes a probability distribution $P(\mathbf{x})$ on $\mathbf{S}$. The utility of each player i is defined as

1. $\tilde{u}_i(\mathbf{x}) = \mathbb{E}_{\mathbf{s} \sim P(\mathbf{x})}[u_i(\mathbf{s})]$,
2. $\tilde{u}_i(\mathbf{x}) = \mathbb{E}_{\mathbf{s} \sim P(\mathbf{x})}[u_i(\mathbf{s})] + \mathbb{V}_{\mathbf{s} \sim P(\mathbf{x})}[u_i(\mathbf{s})]$, where $\mathbb{V}$ denotes the variance,
3. $\tilde{u}_i(\mathbf{x}) = \mathbb{E}_{\mathbf{s} \sim P(\mathbf{x})}[u_i(\mathbf{s})] - \mathbb{V}_{\mathbf{s} \sim P(\mathbf{x})}[u_i(\mathbf{s})]$, where $\mathbb{V}$ denotes the variance,
4. $\tilde{u}_i(\mathbf{x}) = \mathbb{P}_{\mathbf{s} \sim P(\mathbf{x})}[u_i(\mathbf{s}) \geq \tau]$ for some constant $\tau \in \mathbb{R}$.

For which of the above definitions is a mixed Nash equilibrium guaranteed to exist? Give short proofs or counterexamples.

Exercise 7.2. Consider bimatrix games with utilities in the unit interval. Show the existence of a mixed strategy profile $(x, y) \in \square(\mathbf{S})$ with $|\mathrm{supp}(x)|, |\mathrm{supp}(y)| \in \{1, 2\}$, $u_1(x, y) \geq u_1(x', y) - \frac{1}{2}$ for all $x' \in \triangle(S_1)$, and $u_2(x, y) \geq u_2(x, y') - \frac{1}{2}$ for all $y' \in \triangle(S_2)$.

Exercise 7.3. Let G be an extensive form game with imperfect information and perfect recall without moves by the artificial (chance) player 0. The game G' differs from G only in the fact that one of the information sets of player 1 in G is split into two information sets. Show that all pure Nash equilibria of G correspond to pure Nash equilibria in G'. Give a counterexample to this statement when there are moves by player 0. Note: In an earlier version, I did not require the game to have perfect recall. You may also solve this exercise by giving an example of a game without perfect recall for which the statement does not hold.

Exercise 7.4. Show that weighted congestion games for which the set of strategies of each player corresponds to the set of bases of a matroid have a pure Nash equilibrium.

Exercise 7.5. For a binary matrix $A \in \{0,1\}^{m \times n}$ and $c \in \mathbb{R}^m$, the packing game is the coalitional game (N, v) with $N = \{1, \ldots, n\}$ and

$$v : 2^N \to \mathbb{R} \qquad S \mapsto \max\{y^\top c : y^\top A \leq \chi(S), y \in \{0,1\}^m\},$$

where $\chi(S)$ is the characteristic vector of the set S, i.e., $\chi(S)_i = 1$ if $i \in S$ and $\chi(S)_i = 0$ otherwise. (We also set $v(\emptyset) = 0$.) Show that this game has a non-empty core if and only if the linear program $\max\{y^\top c : y^\top A \leq \chi(N), y \geq 0\}$ has an integer optimal solution.

Exercise 7.6. Let $G = (V, E)$ be a graph with flow-dependent cost functions on the edges and commodities $(u_i, v_i, d_i) \in V \times V \times \mathbb{R}_{>0}$, $i = 1, \ldots, k$ and let f be a Wardrop flow for G. Let g be a system optimum for the commodities $(u_i, v_i, 2d_i)$, $i = 1, \ldots, k$ with twice the demands. Show that $C(f) \leq C(g)$.

This is an extra assignment sheet. Solutions to this sheet count towards the total number of 24 exercises that have to be solved. There will be no exercise session for this sheet. All solutions have to be submitted by February 10, 12:00 via email.