Algorithmic Game Theory

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Assignment Sheet 4

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Exercise 4.1. Consider an n-player game where players correspond to governments of countries. Every government has two strategies: enforcing a reduction of the emission of greenhouse gas, or doing nothing. Reducing emissions increases the private cost of that country by 3, doing nothing increases the private cost of <u>all</u> countries by 1. Give an exact potential of this game and determine its pure Nash equilibria.

Exercise 4.2. A function $P : \mathbf{S} \to \mathbb{R}$ is a weighted potential of the strategic game $G = (N, \mathbf{S}, \boldsymbol{\pi})$ if there is a vector $(w_i)_{i \in N}$ with $w_i > 0$ such that $P(t_i, \mathbf{s}_{-i}) - P(\mathbf{s}) = w_i\big(\pi_i(t_i, \mathbf{s}_{-i}) - \pi_i(\mathbf{s})\big)$ for all $i \in N$, $\mathbf{s} \in \mathbf{S}$, $t_i \in S_i$. Show that a finite game G has a weighted potential if and only if its mixed extension has a weighted potential.

Exercise 4.3. A strategic game $G = (N, \mathbf{S}, \boldsymbol{\pi})$ is called best-reply potential game if there is a function $P : \mathbf{S} \to \mathbb{R}$ such that for all $i \in N$ und $\mathbf{s}_{-i} \in \mathbf{S}_{-i}$ we have $\arg\min_{s_i \in S_i} \pi_i(s_i, \mathbf{s}_{-i}) = \arg\min_{s_i \in S_i} P(s_i, \mathbf{s}_{-i})$. Show that a best-reply potential game G has a pure Nash equilibrium if $\mathbf{S}$ is finite, or S_i is compact and P is continuous.

Exercise 4.4. Let $G = (N, \mathbf{S}, \boldsymbol{\pi})$ be a strategic game and let $\gamma = (\mathbf{s}^0, \mathbf{s}^1, \dots, \mathbf{s}^L = \mathbf{s}^0)$ be a finite and closed sequence of length L in $\mathbf{S}$ such that for each $l \in \{1, \dots, L\}$, there is a player i_l with $\mathbf{s}^{l-1}_{-i_l} = \mathbf{s}^{l}_{-i_l}$ and $s^{l-1}_{i_l} \neq s^{l}_{i_l}$. Let $I(\gamma) = \sum_{l=1}^{L}\big(\pi_{i_l}(\mathbf{s}^l) - \pi_{i_l}(\mathbf{s}^{l-1})\big)$. Show that the following are equivalent:

1. G is an exact potential game.
2. $I(\gamma) = 0$ for every finite and closed sequence γ.
3. $I(\gamma) = 0$ for every finite and closed sequence γ of length 4.

Exercise 4.5. Consider a graph $D = (V, E)$ with constant edge cost $k_e > 0$ and a set of n players where a strategy for each player i is to choose a path between two designated nodes $u_i, v_i \in V$. Edge costs are distributed evenly between all players choosing them and the private cost of each player i is the sum of their shares of the costs. Show that this game is an unweighted congestion game and give its potential function. For a strategy profile $\mathbf{s}$, let $C(\mathbf{s}) = \sum_{i \in N} \pi_i(\mathbf{s})$ denote its social cost. Show that there is a pure Nash equilibrium $\mathbf{s}^*$ with $C(\mathbf{s}^*) \leq H_n \cdot \min_{\mathbf{s} \in \mathbf{S}} C(\mathbf{s})$, where H_n is the nth harmonic number.

Exercise 4.6. Consider the pentagon in which vertices are numbered from 0 to 4 and each vertex $i \in \{0, \dots, 4\}$ is connected by an edge with the vertices $(i+1) \mod 5$ and $(i-1) \mod 5$. Every vertex i has a weight $w_i \in \mathbb{Z}$ such that $\sum_{i=0}^{4} w_i > 0$. Consider the algorithm that, in each step, chooses a vertex i with negative weight, adds w_i to the weight of its neighbors, and then multiplies w_i with -1. Show that after finitely many iterations all weights are non-negative.