A Cooperative Games

So far we have considered non-cooperative games, in which each players acts on its own. The theory of cooperative games, on the other hand, studies which coalitions of players are likely to form in a given situation. It takes the payoff achievable by any coalition as given, but requires that coalitions can distribute these payoffs among their members in such a way that the members are satisfied. We consider games with transferable payoff, where the payoff obtained by a coalition can be distributed in an arbitrary way among its members, and restrict our attention to the grand coalition N consisting of all players.

Formally, a *coalitional game* is given by a set $N = \{1, \dots, n\}$ of *players*, and a *characteristic function* $v : 2^N \to \mathbb{R}$ that maps each coalition of players to its *value*, the joint payoff the coalition can obtain by working together. For a given game (N, v), a vector $x \in \mathbb{R}^n$ of payoffs is said to satisfy *(economic) efficiency* if $\sum_{i \in N} x_i = v(N)$ and *individual rationality* if $x_i \geqslant v(\{i\})$ for $i = 1, \dots, n$. The first condition intuitively ensures that no payoff is wasted, while the second condition ensures that each player obtains at least the same payoff it would be able to obtain on its own. A payoff vector that is both efficient and individually rational is also called an *imputation*.

A.1 The Core

Efficiency and individual rationality may not be enough to guarantee a stable outcome. For any two imputations x and y, $\sum_{i \in N} x_i = \sum_{i \in N} y_i = v(N)$, so $y_i > x_i$ for some $i \in N$ implies that $y_j < x_j$ for some other $j \in N$. However, there could be some coalition $S \subseteq N$ such that $y_i > x_i$ for all $i \in S$. If in addition $\sum_{i \in S} y_i \leqslant v(S)$, the members of S could increase their respective payoffs by deviating from the grand coalition, forming the coalition S, and distributing the payoff thus obtained according to y. The core is the set of imputations that are stable against this kind of deviation. Formally, imputation x is in the core of game (N, v) if $\sum_{i \in S} x_i \geqslant v(S)$ for all $S \subseteq N$.

Consider a situation where $n \geqslant 2$ members of an expedition have discovered a treasure, and any pair of them can carry one piece of the treasure back home. This situation can be modeled by a coalitional game (N, v) where $N = \{1, \dots, n\}$ and $v(S) = |S|/2$ if $|S|$ is even and $v(S) = (|S| - 1)/2$ if $|S|$ is odd. The core then contains all imputations if $n = 2$, the single imputation $(1/2, \dots, 1/2)$ if $n \geqslant 4$ is even, and is empty if n is odd. The latter can for example be shown using a characterization of games with a non-empty core, which we discuss next.

Call a function $\lambda : 2^N \to [0, 1]$ *balanced* if for every player the weights of all coalitions containing that player sum to 1, i.e., if for all $i \in N$, $\sum_{S \subseteq N \setminus \{i\}} \lambda(S \cup \{i\}) = 1$. A game (N, v) is called *balanced* if for every balanced function λ, $\sum_{S \subseteq N} \lambda(S) v(S) \leqslant v(N)$. The intuition behind this definition is that each player allocates one unit of time among the coalitions it is a member of, and each coalition earns a fraction of its value proportional to the minimum amount of time devoted to it by any of its members. Balancedness of a collection of weights imposes a feasibility condition on players' allocations of time, and a game is balanced if there is no feasible allocation that yields more than $v(N)$.

THEOREM A.1 (Bondareva 1963, Shapley 1967). *A game has a non-empty core if and only if it is balanced.*

Proof. The core of a game (N, v) is non-empty if and only if the linear program to

$$\begin{aligned} \text{minimize} \quad & \sum_{i \in N} x_i \\ \text{subject to} \quad & \sum_{i \in S} x_i \geqslant v(S) \quad \text{for all } S \subseteq N \end{aligned}$$

has an optimal solution with value $v(N)$. This linear program has the following dual:

$$\begin{aligned} \text{maximize} \quad & \sum_{S \subseteq N} \lambda(S) v(S) \\ \text{subject to} \quad & \sum_{S \subseteq N, i \in S} \lambda(S) = 1 \quad \text{for all } i \in N \\ & \lambda(S) \geqslant 0 \quad \text{for all } S \subseteq N, \end{aligned}$$

where $\lambda : 2^N \to \mathbb{R}$. Note that λ is feasible for the dual if and only if it is a balanced function. Both primal and dual are feasible, so by strong duality their optimal objective values are the same. This means that the core is non-empty if and only if $\sum_{S \subseteq N} \lambda(S)v(S) \leqslant v(N)$ for every balanced function λ. □

To see that the core of our example game is empty if n is odd, define $\lambda : 2^N \to [0, 1]$ such that $\lambda(S) = 1/(n-1)$ if $|S| = 2$ and $\lambda(S) = 0$ otherwise. Then, for all $i \in N$, $\sum_{S \subseteq N \setminus \{i\}} \lambda(S \cup \{i\}) = 1$, because each player is contained in exactly $(n-1)$ sets of size 2. Moreover, $\sum_{S \subseteq N} \lambda(S)v(S) = n(n-1)/2 \cdot 1/(n-1) = n/2$, which is greater than $v(N)$ if n is odd.

A.2 The Shapley Value

The core is problematic as a solution concept because it may be empty or contain a large number of elements, and may thus fail to provide any kind of prediction or guidance. Shapley took a different approach to the distribution of joint payoff among the members of a coalition, by considering solution concepts that yield a single payoff vector for each game as well as a number of axioms that a good solution concept should satisfy.

Fix a set N of players and let a solution concept for games with this set of players be given by a function $\phi : \mathbb{R}^{Q(N)} \to \mathbb{R}^N$, where $Q(N) = 2^N \setminus \{\emptyset\}$. To define Shapley's axioms we need some notation. Given a game (N, v) and a permutation $\pi : N \to N$ of the set of players, let (N, v_π) be the game obtained from (N, v) when for all $i \in S$, $\pi(i)$ takes the role of i, such that $v_\pi(\{\pi(i) : i \in S) = v(S)$ for all $S \subseteq N$. Call $C \subseteq N$ a carrier of game (N, v) if it contains all players that contribute to the value of some coalition, i.e., if $v(S \cap N) = v(S)$ for all $S \subseteq N$. Then a solution concept ϕ is called

- *symmetric* if for every $v \in \mathbb{R}^{Q(N)}$, $\pi : N \to N$, and $i \in N$, $\phi_{\pi(i)}(v_\pi) = \phi_i(v)$;
- *carrier-consistent* if for every $v \in \mathbb{R}^{Q(N)}$ and every carrier C of the game (N, v), $\sum_{i \in C} \phi_i(v) = v(C)$; and
- *linear* if for every $v, w \in \mathbb{R}^{Q(N)}$, $p \in [0, 1]$, and $i \in N$, $\phi_i(pv + (1-p)w) = p\phi_i(v) + (1-p)\phi(w)$.

The intuition behind these axioms is that players who contribute the same should receive the same payoff, that the overall payoff should be shared precisely among players who contribute to some coalition, and that payoffs for a lottery over two games should be equal to the expected payoffs of choosing a game according to the lottery and playing that game. Quite surprisingly there is exactly one solution concept that satisfies the axioms, the so-called *Shapley value* given by

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \big(v(S \cup \{i\}) - v(S)\big). \tag{A.1}$$

The Shapley value of player i can be interpreted as its average contribution over all possible orders in which the players can join the grand coalition.

Consider for example the game with $N = \{1, 2, 3\}$ and characteristic function v given by

$$v(\{1\}) = 1 \quad v(\{2\}) = 2 \quad v(\{3\}) = 1$$
$$v(\{1,2\}) = 2 \quad v(\{1,3\}) = 3 \quad v(\{2,3\}) = 5 \quad v(\{1,2,3\}) = 4.$$

Then

$$\begin{aligned}
\phi_1(v) &= \frac{0!2!}{3!}(v(\{1\}) - v(\emptyset)) + \frac{1!1!}{3!}(v(\{1,2\}) - v(\{2\})) + \\
&\quad \frac{1!1!}{3!}(v(\{1,3\}) - v(\{3\})) + \frac{2!0!}{3!}(v(\{1,2,3\}) - v(\{2,3\})) \\
&= \frac{1}{3}\,1 + \frac{1}{6}\,0 + \frac{1}{6}\,2 + \frac{1}{3}\,(-1) = \frac{1}{3}, \\
\phi_2(v) &= \frac{0!2!}{3!}(v(\{1\}) - v(\emptyset)) + \frac{1!1!}{3!}(v(\{1,2\}) - v(\{2\})) + \\
&\quad \frac{1!1!}{3!}(v(\{1,3\}) - v(\{3\})) + \frac{2!0!}{3!}(v(\{1,2,3\}) - v(\{2,3\})) = \frac{11}{6}, \quad \text{and} \\
\phi_3(v) &= \frac{0!2!}{3!}(v(\{1\}) - v(\emptyset)) + \frac{1!1!}{3!}(v(\{1,2\}) - v(\{2\})) + \\
&\quad \frac{1!1!}{3!}(v(\{1,3\}) - v(\{3\})) + \frac{2!0!}{3!}(v(\{1,2,3\}) - v(\{2,3\})) = \frac{11}{6}.
\end{aligned}$$

THEOREM A.2 (Shapley, 1953). *The Shapley value is the unique solution concept that is symmetric, carrier-consistent, and linear.*

Proof. It is easy to see that ϕ as defined in (A.1) satisfies the axioms: it is linear in ν; it is symmetric because each coefficient depends only on the size of coalition S; and it is carrier-consistent because $\sum_{i \in N} \phi_i(\nu) = \nu(N)$, and $\phi_i(\nu) = 0$ if i contributes zero to every coalition $S \subseteq N \setminus \{i\}$.

We now consider a solution concept ψ that satisfies the axioms and show that it must be unique. Linearity implies that ψ is an affine function from $\mathbb{R}^{Q(N)}$ to $\mathbb{R}^N$. Moreover, by carrier-consistency, $\psi_i(z) = 0$ for all $i \in N$ and the game (N, z) with $z(S) = 0$ for all $S \subseteq N$. This means that ψ is in fact a linear function from $\mathbb{R}^{Q(N)}$ to $\mathbb{R}^N$. We proceed by constructing a set of games for which the behavior of ψ is completely determined by the axioms. We then show that this set forms a basis of the vector space $\mathbb{R}^{Q(N)}$ and use the fact that the behavior of any linear function on a vector space is completely determined by its behavior on a basis.

For any $C \in Q(N)$, let $w_C : \mathbb{R}^{Q(N)} \to \mathbb{R}$ be the function with $w_C(S) = 1$ if $C \subseteq S$ and $w_C(S) = 0$ otherwise. Then, by carrier-consistency, $\sum_{i \in C} \psi(w_C) = 1$, and $\psi(w_C) = 0$ if $i \notin C$. In fact, by symmetry, $\psi_i(w_C) = 1/|C|$ if $i \in C$ and $\psi_i(w_C) = 0$. There are $2^{|N|} - 1$ such games w_C, which is equal to the dimension of $\mathbb{R}^{Q(N)}$. We claim moreover that they are linearly independent in $\mathbb{R}^{Q(N)}$, i.e., that for any $\alpha \in \mathbb{R}^{Q(N)}$, $\sum_{C \in Q(N)} \alpha_C w_C = 0$ implies that $\alpha_C = 0$ for all $C \in Q(N)$. Assume this was not the case and consider $S \subseteq N$ of minimum cardinality such that $\alpha_C \neq 0$; then $\alpha_S = \sum_{C \subseteq S, C \neq \emptyset} \alpha_C = \sum_{C \in Q(N)} \alpha_C w_C(S) = 0$, a contradiction. Thus $\{w_C : C \in Q(N)\}$ is a basis of $\mathbb{R}^{Q(N)}$. Since ψ is a linear function on $\mathbb{R}^{Q(N)}$ its behavior is determined completely by its behavior on this basis, which in turn is unique. $\square$