Chapter 1

Games and Their Equilibria

The central notion of game theory that captures many aspects of strategic decision making is that of a strategic game.

Definition 1.1 (Strategic Game). An n-player strategic game is a triplet $G = (N, \mathbf{S}, \mathbf{u})$, where

- N is a finite and non-empty set of n players,
- $\mathbf{S} = \bigtimes_{i \in N} S_i$ is the set of strategy vectors $\mathbf{s} = (s_i)_{i \in N}$ and S_i is the set of strategies available to player i,
- $\mathbf{u} : \mathbf{S} \to \mathbb{R}^n$, $\mathbf{s} \mapsto \bigtimes_{i \in N} u_i(\mathbf{s})$ is the combined utility function and $u_i(\mathbf{s})$ is the utility of player i when strategy profile $\mathbf{s}$ is played.

The fact that the utility of player i depends on the whole vector $\mathbf{s}$ rather than their own strategy s_i only, distinguishes a strategic game from an ordinary optimization problem.
In this chapter, we use the convention that vectors of objects that contain one element for each player, are denoted with a bold face. Examples are the joint strategy space $\mathbf{S} = \bigtimes_{i \in N} S_i$, strategy vectors $\mathbf{s} = (s_i)_{i \in N}$ and the combined utility function $\mathbf{u} : \mathbf{S} \to \mathbb{R}$.
A strategy profile from which no player has an incentive to deviate unilaterally is called a Nash equilibrium. For a formal definition, we introduce the following notation. For a strategy profile $\mathbf{s} \in \mathbf{S}$, a player i, and an alternative strategy $k \in S_i$, we write

$$(k, \mathbf{s}_{-i}) = (s_1, \dots, s_{i-1}, k, s_{i+1}, \dots, s_n)$$

for the strategy profile in which all players play as in $\mathbf{s}$, except for player i who plays k. We also write $\mathbf{S}_{-i}$ for the space of strategy profiles of all players except i.

Definition 1.2 (Nash Equilibrium). A strategy profile $\mathbf{s}$ is a Nash equilibrium if

$$u_i(\mathbf{s}) \geq u_i(k, \mathbf{s}_{-i})$$

for all $i \in N$ and $k \in S_i$.

	$\dots$	l	$\dots$	l'	$\dots$
$\vdots$		$\vdots$		$\vdots$	
k	$\dots$	$(u_1(k,l), u_2(k,l))$	$\dots$	$(u_1(k,l'), u_2(k,l'))$	$\dots$
$\vdots$		$\vdots$		$\vdots$	
k'	$\dots$	$(u_1(k',l), u_2(k,l))$	$\dots$	$(u_1(k',l'), u_2(k',l'))$	$\dots$
$\vdots$		$\vdots$		$\vdots$	

Figure 1.1: Bi-matrix representation of a finite two-player game in strategic form.

Another way to look at a Nash equilibrium is to say that each player i plays best against the strategy vector $\mathbf{s}_{-i}$ of the other players. To formalize this viewpoint, we define best replies.

Definition 1.3 (Best Reply). The set-valued function $\beta_i : \mathbf{S} \to 2^{S_i}$ defined as

$$\beta_i(\mathbf{s}) = \arg\max_{k \in S_i} u_i(k, \mathbf{s}_{-i})$$

is called best-reply function of player i.

Setting $\boldsymbol{\beta}(\mathbf{s}) = \bigtimes_{i \in N} \beta_i(\mathbf{s})$, we obtain the following immediate proposition.

Proposition 1.4. A strategy profile $\mathbf{s}$ is a Nash equilibrium if and only if $\mathbf{s} \in \boldsymbol{\beta}(\mathbf{s})$.

1.1 Finite Games

A game $G = (N, \mathbf{S}, \mathbf{u})$ is called finite, if $\mathbf{S}$ is finite. Finite two player games are usually represented as a table of pairs of utilities, see Figure 1.1. The strategies $k \in S_1$ of player 1 correspond to the rows of the table, and the strategies $l \in S_2$ correspond to its columns. The entry in row k and column l is the pair $(u_1(k,l), u_2(k,l))$.
We proceed to discuss a few classic two-player games.

Example 1.5 (Bach or Stravinsky). Two people want to visit a concert together. Player 1 prefers Bach over Stravisnky; player 2 prefers Stravinsky over Bach. Both receive a utility of 0 when being without the friend in the concert. A realization of this game is given in Figure 1.2a. The game has two Nash equilibria, the strategy profiles $(\text{Bach}, \text{Bach})$ and $(\text{Stravinsky}, \text{Stravinsky})$ shown as coloured boxes in Figure 1.2a.

	Bach	Stravinsky
Bach	(2, 1)	(0, 0)
Stravinsky	(0, 0)	(1, 2)

(a) Bach-or-Stravinsky

	Mozart	Mahler
Mozart	(2, 2)	(0, 0)
Mahler	(0, 0)	(1,1)

(b) Mozart-or-Mahler

Figure 1.2: Two simple 2×2 games. **(a)** Bach or Stravinsky game (Example 1.5); **(b)** Mozart-or-Mahler game (Example 1.6).

	Confess	Silence
Confess	(−3, −3)	(0, −4)
Silence	(−4, 0)	(−1, −1)

Figure 1.3: The Prisoner's Dilemma (Example 1.7).

Example 1.6 (Mozart or Mahler). A game with $u_i(s) = u_j(s)$ for all $i, j \in N$ and $s \in S$ is called a coordination game. As an example, consider the Mozart-or-Mahler game that takes place in the same context as the Bach-or-Stravinsky game Figure 1.2b with the only difference that now both players prefer Mozart over Mahler. The game has the two Nash equilibria (Mozart, Mozart) and (Mahler, Mahler). Note that (Mahler, Mahler) is a Nash equilibrium, even though both players prefer Mozart over Mahler.

Example 1.7 (Prisoner's Dilemma). Two suspects in a crime are interrogated by the police. Evidences gathered by the police are sufficient to sentence both of them for one year in prison. Both are offered separately the possibility to confess their crimies. If only one of them confesses, that person will be freed and the other one will be sentenced to 4 years in prison. If both confess, they both will spend 3 years in prison. Both criminals strive to minimize their years in prison. A representation in terms of a utility maximization game is given in Figure 1.3. Although both players are better off in the strategy profile (Silence, Silence), the unique Nash equilibrium of the game is (Confess, Confess).

Example 1.8 (Matching Pennies). Each of two players chooses either Head or Tail. Player 1 wins if both choose the same strategy, otherwise player 2 wins. Both players want to win the game. A representation of this situation is shown in Figure 1.4a. This game has no Nash equilibrium.

	Head	Tail
Head	$(1,-1)$	$(-1,1)$
Tail	$(-1,1)$	$(1,-1)$

(a) Original game.

	Head	Tail	Random
Head	$(1,-1)$	$(-1,1)$	$(0,0)$
Tail	$(-1,1)$	$(1,-1)$	$(0,0)$
Random	$(0,0)$	$(0,0)$	$(0,0)$

(b) After adding the Random strategy.

Figure 1.4: Matching Pennies (Example 1.8). **(a)** Original game. **(b)** After adding the strategy Random for both players which allows to play one of the original strategies uniformly at random.

Let us add a new strategy, termed Random, to the set of strategies of both players. This strategy allows each player to play one of their original games Head and Tail uniformly at random. Let us further assume that the utility received when at least one of the players plays the Random strategy is equal to the expected utility. Then, we arrive at the finite game in Figure 1.4b. The unique Nash equilibrium of this game is (Random, Random).

The Matching Pennies Game illustrates that randomization may help to make sensible predictions on the players' behavior when no equilibrium in deterministic exists. Actually, one may argue that the whole point of games like the Matching Pennies Game lies in the fact that player need to randomize and, thus, the outcome of the game cannot be foreseen.
The possibility of randomization changes the strategy set of each player and, thus, gives rise to a new game that is called the mixed extension.

1.2 Mixed Extensions of Finite Games

A mixed strategy of player i in the finite strategic game $G = (N, \mathbf{S}, \mathbf{u})$ is a probability distribution over their set S_i of pure strategies. Formally, for a finite set S_i of $m_i = |S_i|$ strategies, let

$$\Delta(S_i) = \left\{ (x_{ik})_{k \in S_i} \in \mathbb{R}^{m_i}_{\geq 0} : \sum\nolimits_{k \in S_i} x_{ik} = 1 \right\}$$

denote the unit simplex spanned by the elements in S_i. A mixed strategy of player i is an element $x_i \in \Delta(S_i)$. For a mixed strategy $x_i = (x_{ik})_{k \in S_i} \in \Delta(S_i)$ and a pure strategy $k \in S_i$, we denote by x_{ik} the stochastic weight that player i puts on k in the mixed strategy x_i. A mixed strategy vector is a vector $\mathbf{x} = (x_i)_{i \in N}$ in the space $\square(\mathbf{S}) = \bigtimes_{i \in N} \Delta(S_i)$.
The mixed extension of a finite game $G = (N, \mathbf{S}, \mathbf{u})$ is the strategic game $\tilde{G} = (N, \tilde{\mathbf{S}}, \tilde{\mathbf{u}})$ with the set of strategy profiles $\tilde{\mathbf{S}} = \square(\mathbf{S})$, and the utility of a player is

the weighted sum of its utilities $u_i(\boldsymbol{s})$ in pure-strategy profiles $\boldsymbol{s} \in \boldsymbol{S}$ weighted by the probability that they occur.

Definition 1.9 (Mixed Extension). Let $G = (N, \boldsymbol{S}, \boldsymbol{u})$ be a finite game. Then, the game $\tilde{G} = (N, \square(\boldsymbol{S}), \tilde{\boldsymbol{u}})$ with the utility functions

$$\tilde{u}_i(\boldsymbol{x}) = \sum_{\boldsymbol{s} \in \boldsymbol{S}} \Big(u_i(\boldsymbol{s}) \cdot \prod\nolimits_{j \in N} x_{js_j} \Big) \tag{1.1}$$

is called the mixed extension of G.

It is convenient to rewrite the utility of player i in the following way.

$$\begin{aligned} \tilde{u}_i(\boldsymbol{x}) &= \sum_{\boldsymbol{s} \in \boldsymbol{S}} \Big(u_i(\boldsymbol{s}) \cdot \prod\nolimits_{j \in N} x_{js_j} \Big) \\ &= \sum_{k \in S_i} \sum_{\boldsymbol{s} \in \boldsymbol{S}: s_i = k} \Big(u_i(\boldsymbol{s}) \cdot \prod\nolimits_{j \in N} x_{js_j} \Big) \\ &= \sum_{k \in S_i} x_{ik} \sum_{\boldsymbol{s} \in \boldsymbol{S}: s_i = k} \Big(u_i(\boldsymbol{s}) \cdot \prod\nolimits_{j \in N \setminus \{i\}} x_{js_j} \Big) \\ &= \sum_{k \in S_i} x_{ik} \cdot \tilde{u}_i(e_{ik}, \boldsymbol{x}_{-i}), \end{aligned} \tag{1.2}$$

where e_{ik} is the degenerate mixed strategy that puts all weight on the pure strategy $k \in S_i$. When expressing the utility of player i as in (1.2), we observe that for a fixed strategy vector $\boldsymbol{x}_{-i} \in \bigtimes_{j \in N \setminus \{i\}} \Delta(S_j)$ the utility of player i is linear in the probabilities x_{ik} that player i puts on the pure strategies $k \in S_i$. Thus, the utility of player i is a linear combination of the utilities obtained from playing its pure strategies weighted by the stochastic weight attached to them in the chosen mixed strategy. This observation gives rise to the following useful lemma concerning the geometry of best replies.

Lemma 1.10. In the mixed extension of a finite game, best replies are sub-simplices spanned by a subset of the pure strategies, i.e., $\beta_i(\boldsymbol{x}) = \Delta(T_i)$ for some $T_i \subseteq S_i$ for all $i \in N$ and $\boldsymbol{x} \in \square(\boldsymbol{S})$.

Proof. Let $\boldsymbol{x} \in \square(\boldsymbol{S})$ and $i \in N$ be arbitrary and let $y_i \in \beta_i(\boldsymbol{x})$. By (1.2), all $k \in \operatorname{supp}(y_i)$ are best replies to $\boldsymbol{x}$ as well. Further, by linearity of (1.2), convex combinations of best replies are best replies. □

Using this lemma together with the characterization of Nash equilibria via best replies (cf. Proposition 1.4), we obtain the following corollary.

Corollary 1.11. A mixed strategy profile $\boldsymbol{x}$ is a mixed Nash equilibrium if and only if $e_{ik} \in \beta_i(\boldsymbol{x})$ for every player i and every pure strategy $k \in \text{supp}(x_i)$.

Proof. Recall from Proposition 1.4 that $\boldsymbol{x}$ is a mixed Nash equilibrium if and only if $\boldsymbol{x} \in \beta(\boldsymbol{x})$, i.e., $x_i \in \beta_i(\boldsymbol{x})$ for all players i. We obtain

$$\begin{aligned} & x_i \in \beta_i(\boldsymbol{x}) \\ \Leftrightarrow \quad & x_i \in \Delta(T_i), \text{ where } T_i = \{k \in S_i : e_{ik} \in \beta_i(\boldsymbol{x})\} \\ \Leftrightarrow \quad & \text{supp}(x_i) \subseteq T_i, \end{aligned}$$

proving the claimed result. □

We obtain as a direct corollary that pure Nash equilibria persist in the mixed extension of a game.

Corollary 1.12. Let $\boldsymbol{s} \in \boldsymbol{S}$ be a Nash equilibrium of the finite game G. Then, the mixed strategy profile $\boldsymbol{x} \in \square(\boldsymbol{S})$ with $x_{is_i} = 1$ for all players i is a Nash equilibrium of $\tilde{G}$.

Example 1.13 (Bach or Stravinsky, Mixed Extension). We reconsider the Bach-or-Stravinsky game in Figure 1.2a. In Example 1.5 we have seen that this game has two pure Nash equilibria, i.e., $(\text{Bach}, \text{Bach})$ and $(\text{Stravinsky}, \text{Stravinsky})$. These are the only equilibria in which player 1 does not mix. For a player i, let $x_{i,B}$ and $x_{i,S}$ denote the probabilities of player i to play Bach and Stravinsky, respectively. Suppose there is a Nash equilibrium $\big((x_{1,B}, x_{1,S}), (x_{2,B}, x_{2,S})\big)$ with $x_{1,B} \in (0, 1)$. Then, by Corollary 1.11, it is necessary that both pure strategies of player 1 yield the same utility, i.e.,

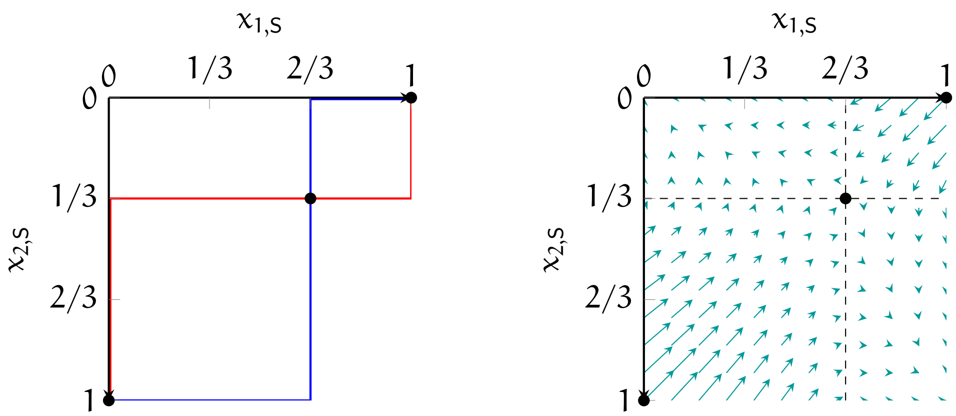
$$2x_{2,B} = 1x_{2,S}$$

Using that $x_2 = (x_{2,B}, x_{2,S})$ is stochastic, this is equivalent to $x_{2,B} = 1/3$ and $x_{2,S} = 2/3$. Again, when player 2 mixes between their two pure strategies in an equilubrium, it is necessary that they yield the same utility, i.e.,

$$1x_{1,B} = 2x_{1,S}$$

which implies $x_{1,B} = 2/3$ and $x_{1,S} = 1/3$. in conclusion, the game has a fully mixed Nash equilibrium where players 1 and 2 play their first strategy with probability 2/3 and 1/3, respectively.

We proceed to prove the important result that every finite game has a mixed Nash equilibrium. Before we do so, we first want to get a glimpse of the analytical tools that can be used to prove such a result. To this end, consider the best-reply functions of the two players in the mixed extension of the Bach-or-Stravinsky



(a) Best reply functions

(b) Nash field

Figure 1.5: Best reply function and Nash field of the Bach-or-Stravinsky game (Example 1.13). **(a)** Best replies of player 1 (blue) and player 2 (red) as a function of the strategy of the other player; **(b)** The function f defined in the proof of Theorem 1.14. Every point (x, y) of the 2-dimensional unit cube corresponds to the mixed strategy vector $\mathbf{x}$ in which player 1 their first strategy with probability x and player 2 plays their first strategy with probability y. Arrows point to the direction to which (x, y) is mapped by f. The three points with vanishing arrows are fixed points of f and, thus, mixed Nash equilibria of the game.

Game shown in Figure 1.5a. As calculated above, player 2 is indifferent between their two pure strategies when player 1 plays $x_{1,B} = 2/3$. Put differently, the best reply of player 2 to the mixed strategy $(2/3, 1/3)$ comprises the whole set $\Delta(S_2)$ of mixed strategies of player 2. Below this threshold value, player 2 favors playing the pure strategy Stravinsky; above this value, they prefer Bach. The points were the best reply functions of both players meet are the Nash equilibria of the game.
This suggest to use a fixed point theorem for set-valued functions to prove the existence of a pure Nash equilibrium. Indeed, this road has been taken in Nash (1950b) where the existence of a Nash equilibrium for mixed extensions of finite games was proven first.
We here instead use the road taken by Nash (1950a), where the problem of finding a Nash equilibrium is formulated as a fixed point of a regular (single-valued) function.

> **Theorem 1.14** (Nash, 1950b). Every mixed extension of a finite game has a Nash equilibrium.

For the proof, we use the following result.

Theorem 1.15 (Brouwer Fixed Point Theorem). Let $f : X \to X$ be a continuous function, where S is closed, bounded and convex. Then, f has a fixed point.

Proof of Theorem 1.14. Let $\tilde{G} = (N, \square(\mathbf{S}), \tilde{u})$ be the mixed extension of a finite games. For a mixed strategy profile $\mathbf{x} \in \square(\mathbf{S})$ and a player i, let $h_{ik}(\mathbf{x})$ denote the excess utility of player i when playing their kth pure strategy instead of x_i, i.e.,

$$h_{ik} = \max\{0, u_i(e_{ik}, \mathbf{x}_{-i}) - u_i(\mathbf{x})\},$$

where again e_{ik} denotes the mixed strategy of player i that puts all weight on the kth pure strategy. Further, define $f : \square(\mathbf{S}) \to \square(\mathbf{S})$ as

$$\mathbf{x} \mapsto \times_{i \in N} (x'_{ik})_{k \in S_i}, \quad \text{where} \quad x'_{ik} = \frac{x_{ik} + h_{ik}(\mathbf{x})}{1 + \sum_{l \in S_i} h_{il}(\mathbf{x})}.$$

Note that all x'_{ik} are non-negative and $\sum_{k \in S_i} x'_{ik} = 1$. Further, function f is continuous and $\square(S)$ is closed, bounded and convex, so it has a fixed point $\mathbf{x}$ with $f(\mathbf{x}) = \mathbf{x}$, i.e.

$$x_{ik} \sum_{k \in S_i} h_{ik}(\mathbf{x}) = h_{ik}(\mathbf{x})$$

for all players i and pure strategies k. If $\sum_{k \in S_i} h_{ik}(\mathbf{x}) > 0$ for a player i, then $h_{ik}(\mathbf{x}) > 0$ for all $k \in S_i$ with $x_{ik} > 0$. This is impossible since not all of the strategies of player i in the support of x_i can grant them above average returns. □

The map f is sometimes called Nash field. A graphical illustration of the Nash field for the mixed extension of the Bach-or-Stravinsky Game (Example 1.5) can be found in Figure 1.5b.

1.3 Euclidean Games

A game $G = (N, \mathbf{S}, \mathbf{u})$ is called Euclidean, if $S_i \subseteq \mathbb{R}^{m_i}$ for all players i and some player-specific dimension $m_i \in \mathbb{N}$.

Example 1.16 (Cournot Oligopoly). Two firms are producing a homogeneous good without production cost. Both firms simultaneously choose their respective production levels. They sell their output on a single market at the market clearing price p, which is a non-increasing function of the total quantity offered. In this example, we assume a linear market reaction that equals $p(x) = \max\{0, 1 - x\}$ for all $x \geq 0$. The payoff of each firm is the profit from selling their goods. This situation can be modeled as the infinite maximization game $G = (\{1, 2\}, \mathbf{S}, \mathbf{u})$, with $S_1 = S_2 = \mathbb{R}_{\geq 0}$ and

$$u_1(\mathbf{s}) = s_1 \cdot p(s_1 + s_2), \qquad u_2(\mathbf{s}) = s_2 \cdot p(s_1 + s_2).$$

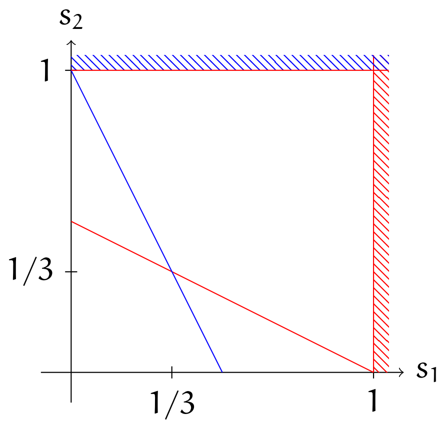


Figure 1.6: Best replies of player 1 (blue) and player 2 (red) as a function of the strategy of the other player in the Cournot oligopoly (Example 1.16).

To find a Nash equilibrium of that game, note that the best reply of player 1 satisfies

$$\beta_1(\mathbf{s}) = \arg\max_{t_1 \in S_1} t_1 \cdot p(t_1 + s_2).$$

If $s_2 < 1$, player 1 can achieve a strictly positive utility by choosing $t_1 < 1 - s_2$. In that case, the first order optimality conditions give

$$\tfrac{\partial}{\partial t_1}\Big(t_1 \cdot p(t_1 + s_2)\Big) = \tfrac{\partial}{\partial t_1}\Big(t_1(1 - t_1 - s_2)\Big) = 1 - 2t_1 - s_2 = 0,$$

which implies $\beta_1(\mathbf{s}) = \frac{1-s_2}{2}$, if $s_2 < 1$. On the other hand, it is easy to see that $\beta_1(\mathbf{s}) = \mathbb{R}_{\geq 0}$, if $s_2 \geq 1$. By symmetry, we obtain the best replies

$$\beta_i(\mathbf{s}) = \begin{cases} \frac{1-s_{-i}}{2}, & \text{if } s_{-i} < 1 \\ \mathbb{R}_{\geq 0}, & \text{else} \end{cases}$$

for all $i \in N$, see Figure 1.6. We obtain two kinds of pure Nash equilibria. In the first type, we have $s_1 < 1$ and $s_2 < 1$, and thus,

$$\frac{1 - s_1}{2} = s_2 = 1 - 2t_1 \qquad \Leftrightarrow \qquad s_1 = s_2 = \frac{1}{3}.$$

There are infinitely other pure Nash equilibria, in which both players choose a strategy greater or equal 1.

Theorem 1.17 (Debreu, 1952; Glicksberg, 1952; Fan, 1952). Let $G = (N, \mathbf{S}, \mathbf{u})$ be an Euclidean game. If each strategy set S_i is non-empty, compact and convex, and each utility function $u_i : \mathbf{S} \to \mathbb{R}$ is continuous and quasi-concave in s_i. Then, G has a Nash equilibrium.

For the proof of this result, we use the following theorem.

Theorem 1.18 (Kakutani, 1941). Let $X \subset \mathbb{R}^n$ be a non-empty, compact and convex set and let $f : X \to 2^X$ be a set-valued function satisfying the following properties:

- f is has a closed graph, i.e.,
  for all $(x_n, y_n)_{n \in \mathbb{N}}$ with $(x_n, y_n) \to (x, y)$ and $y_n \in f(x_n)$ for all n, we have $y \in f(x)$.
- $f(x)$ is non-empty and convex for all $x \in X$.

Then f has a fixed point, i.e., there is $x \in X$ with $x \in f(x)$.

Proof of Theorem 1.17. We will apply Kakutani's Fixed Point Theorem to the joint best-reply function $\boldsymbol{\beta}$. First, we show that $\boldsymbol{\beta}$ satisfies the conditions of Kakutani's theorem.

Step 1: The set of strategy vectors $\mathbf{S}$ is compact, convex, and non-empty since it is the finite Cartesian product of the compact, convex, and non-empty sets S_i.

Step 2: The joint best reply $\boldsymbol{\beta}(\mathbf{s})$ is non-empty for all $\mathbf{s} \in \mathbf{S}$ since for every player i the set $\beta_i(\mathbf{s}) = \arg\max_{t_i \in S_i} u_i(t_i, \mathbf{s}_{-i})$ is non-empty by the continuity of u_i and Weierstrass's theorem.

Step 3: The joint best reply takes only convex values. Clearly, for every player i, the set $\beta_i(\mathbf{s})$ is convex as u_i is concave in s_i. Thus, $\boldsymbol{\beta}(\mathbf{s})$ is convex as the finite Cartesian product of convex sets.

Step 4: For a contradiction, let us assume that $\boldsymbol{\beta}$ does not have a closed graph, i.e., there is a sequence $(\mathbf{x}^n, \mathbf{y}^n)_{n \in \mathbb{N}}$ with $(\mathbf{x}^n, \mathbf{y}^n) \to (x, y)$ and $y^n \in \boldsymbol{\beta}(\mathbf{x}^n)$, but $\mathbf{y} \notin \boldsymbol{\beta}(\mathbf{x})$, i.e., there exists a player i such that $y_i \notin \beta_i(\mathbf{x})$.
This implies the existence of $\epsilon > 0$ and $x_i' \in S_i$ such that

$$u_i(x_i', \mathbf{x}_{-i}) > u_i(y_i, \mathbf{x}_{-i}) + 3\epsilon.$$

By the continuity of u_i and since $\mathbf{x}_{-i}^n \to x_{-i}$, we have for sufficiently large n,

$$u_i(x_i', \mathbf{x}_{-i}^n) \geq u_i(x_i', \mathbf{x}_{-i}) - \epsilon.$$

We conclude

$$u_i(x_i', \mathbf{x}_{-i}^n) > u_i(y_i, \mathbf{x}_{-i}) + 2\epsilon \geq u_i(y_i^n, x_{-i}^n) + \epsilon,$$

where the second relation follows again from the continuity of u_i. This contradicts $y_i^n \in \beta_i(\mathbf{x}^n)$. By Proposition 1.4, every fixed point of the joint best reply function is a Nash equilibrium. □

1.4 Two-Player Zero-Sum-Games

For mixed extensions of two-player zero sum games, the existence of Nash equilibria can be show using much less heavy machinery than Brouwer's or Kakutani's fixed point theorems.

	L	C	R
T	2	0	1
M	2	3	2
B	1	2	0

(a)

	R	P	S
R	0	−1	1
P	1	0	−1
S	−1	1	0

(b)

Figure 1.7: Two two-player zero-sum games (Example 1.21 and Example 1.22).

Definition 1.19 (Two-Player Zero-Sum-Game). A two-player game G is called Zero-Sum Game, if $u_1(\mathbf{s}) = -u_2(\mathbf{s})$ for all $\mathbf{s} \in \mathbf{S}$.

Finite two-player zero-sum games are usually represented by a matrix $A = (a_{kl})$, where the rows of the matrix correspond to strategies of player 1, columns correspond to strategies of player 2, and entries a_{kl} correspond to the utility of player 1 when player 1 plays pure strategy k and player 2 plays pure strategy l. In this setting, player 1 strives to maximize the outcome of the game while player 2 strives to minimize this value.
We are first interested in the strategies of the two players that guarantee them the best outcome.

Definition 1.20 (Maximin Strategy, Minimax Strategy). Let G be a finite two-player zero-sum game. We call
– a strategy $k \in \arg\max_{k \in S_1} \min_{l \in S_2} a_{kl}$ maximin strategy;
– a strategy $l \in \arg\min_{l \in S_2} \max_{k \in S_1} a_{kl}$ minimax strategy.

The utility that a player can guarantee by playing a maximin strategy or minimax strategy is called the security level of that player. We will denote the security level of player i by σ_i.

Example 1.21 (A Two-Player Zero-Sum Game). Consider the game in Figure 1.7(a). The maximin strategy of player 1 is M and their security level $\sigma_1 = 2$. The minima are attained for the strategies L and R of player 2. The Minmax-Strategies of player 2 are L and R guaranteeing player 2 with security level $\sigma_2 = 2$. This game has two Nash equilibria, (M, L) and (M, R).

Example 1.22 (Rock-Paper-Scissors). Consider the Rock-Paper-Scissors game in Figure 1.7(b). Intuitively, all strategies of player 1 are maximin strategyies guaranteeing an outcome of at least -1 and all strategies of player 2 are Minmax-Strategies guranateeing an outcome of at most 1. The game does not have a Nash equilibrium.

For mixed extensions of finite two-player games, the security level of a player and the corresponding maximin strategy or minimax strategy can be computed can be computed straightforwardly by the following linear program:

$$\begin{aligned} &\text{maximize } v && && \text{(P)} \\ &\text{subject to } v - \sum_{i=1}^{m} x_i a_{ij} \leq 0 && \text{for } j = 1, \ldots, n \\ &\sum_{i=1}^{m} x_i = 1 \\ &x_i \geq 0 && \text{for } j = i, \ldots, m. \end{aligned}$$

Theorem 1.23 (von Neumann, 1928). Fix mixed extensions of finite two-player games $\sigma_1 = \sigma_2$.

Proof. Recall that the security level of player 1, σ_1, is the optimal solution of the linear program (P). Its dual is equal to

$$\begin{aligned} &\text{minimize } w && && \text{(D)} \\ &\text{subject to } w - \sum_{j=1}^{n} a_{ij} y_j \geq 0 && \text{for } j = 1, \ldots, n && (1.3) \\ &\sum_{j=1}^{n} y_i = 1 && && (1.4) \\ &y_i \geq 0 && \text{for } i = 1, \ldots, n. && (1.5) \end{aligned}$$

Thus, the solution of the dual program corresponds to a minimax strategy of player 2. Obviously both the primal and the dual linear program have a solution. So, by strong duality, their values coincide. □

Lemma 1.24. For finite two-player zero-sum games, $\sigma_1 \leq \sigma_2$.

Proof. The pure strategies of a player are feasible for the linear programs (P) and (D). By weak duality, $\sigma_1 \leq \sigma_2$. □

Lemma 1.24 holds in fact for arbitrary two-player zero-sum games (for which the notions of maximin and minimax strategies are well-defined). We leave the simple proof of this result to the reader as Exercise 1.32.

The following theorem characterizes the strategies that can occur in a Nash equilibrium of a two-player zero-sum game.

Theorem 1.25. Let G be a finite zero-sum game or its mixed extension. The strategy profile (k^*, l^*) is a Nash equilibrium of G if and only if the following hold:
- k^* is a maximin strategy of player 1,
- l^* is a minimax strategy of player 2,
- $u_1(k^*, l^*) = \sigma_1 = \sigma_2$.

Proof. "$\Rightarrow$": Let (k^*, l^*) be a Nash equilibrium. Since player 2 cannot gain from a deviation to strategy $l \in S_2$,

$$u_1(k^*, l^*) = \min_{l \in S_2} u_1(k^*, l) \leq \max_{k \in S_1} \min_{l \in S_2} u_1(k, l) = \sigma_1. \tag{1.6}$$

Analogously,

$$u_1(k^*, l^*) = \max_{k \in S_1} u_1(k, l^*) \geq \min_{l \in S_2} \max_{k \in S_1} u_1(k, l) = \sigma_2, \tag{1.7}$$

so that we obtain $\sigma_2 \leq u_1(k^*, l^*) \leq \sigma_1$. Together with Lemma 1.24 this implies $\sigma_1 = u_1(k^*, l^*) = \sigma_2$. Thus, inequalities (1.6) and (1.7) are satisfied with equality and k^* and l^* are maximin and minimax strategies, respectively.

"$\Leftarrow$": Let k^* be a maximin strategy of player 1 and l^* be a minimax strategy of player 2 as in the statement of the theorem. As player 1 can guarantee an outcome of at least σ_1, we have

$$u_1(k^*, l^*) = \sigma_1 = \max_{k \in S_1} \min_{l \in S_2} u_1(k, l) = \min_{l \in S_2} u_1(k^*, l) \leq u_1(k^*, l) \tag{1.8}$$

for all $l \in S_2$. Analogously, for player 2,

$$u_1(k^*, l^*) = \sigma_2 = \min_{l \in S_2} \max_{k \in S_1} u_1(k, l) = \max_{k \in S_1} u_1(k, l^*) \geq u_1(k, l^*) \tag{1.9}$$

for all $k \in S_1$, i.e., (k^*, l^*) is a Nash equilibrium with outcome $a_{k^* l^*}$. □

We obtain an alternative proof of the existence of mixed Nash equilibria for finite zero-sum games.

Corollary 1.26. Every mixed extension of a finite zero-sum game has a Nash equilibrium.

Proof. By Theorem 1.23, for every such game $\sigma_1 = \sigma_2$. By Theorem 1.25 a maximin strategy of player 1 together with a minimax strategy of player 2 constitute a Nash equilibrium. □

We further obtain as a direct corollary that all Nash equilibria give the same outcome of the game and that equilibrium strategies can be used interchangeably.

	Hawk	Dove
Hawk	(a, a)	(d, b)
Dove	(b, d)	(c, c)

Figure 1.8: Hawk-Dove-Game

East:

	Left	Right
Top	$(3, 0, 2)$	$(0, 2, 0)$
Bottom	$(0, 1, 0)$	$(1, 0, 0)$

West:

	Left	Right
Top	$(1, 0, 0)$	$(0, 1, 0)$
Bottom	$(0, 1, 0)$	$(2, 0, 3)$

Figure 1.9: Three-player game with unique equilibrium in irrational mixed strategies. Player 1 os the row player choosing between Top and Bottom. Player 2 is the column player with strategies Left and Right. Player 3 chooses between the left utility matrix (East) and the right utility matrix (West). Each cell shows the vector of utilities of players 1, 2, and 3.

Corollary 1.27. Let (k, l) and (k', l') be two Nash equilibria of a two-player zero-sum game. Then, the following holds:
1. $u_1(k, l) = u_1(k', l')$;
2. (k, l') and (k', l) are Nash equilibria as well.

1.5 Exercises

Exercise 1.28. Determine all Nash equilibria of the mixed extension of the Hawk-Dove-Game in Figure 1.8, depending on the parameters $a < b < c < d$.

Exercise 1.29. Show that the three-player game in Figure 1.9 has a unique equilibrium in which all three players play according to an irrational strategy.

Exercise 1.30. Let $\mathcal{G}$ be the class of mixed extensions of finite games with two players and $|S_1| = |S_2| = 2$. Determine the set

$$\mathcal{N} = \{n \in \mathbb{N} \cup \{0, \infty\} : \text{there is } G \in \mathcal{G} \text{ with exactly } n \text{ Nash equilibria}\}.$$

Exercise 1.31. A finite two-player game $G = (\{1, 2\}, \mathbf{S}, \mathbf{u})$ with utility matrices A and B is called symmetric if $A = B^{\top}$. A mixed strategy profile (x, y) is symmetric if $x = y$ and an equilibrium is symmetric if it is a symmetric strategy profile. Show that mixed extensions of symmetric games have a symmetric Nash equilibrium.

Exercise 1.32. Let G be a two-player zero-sum game with compact strategy sets S_i for $i = 1, 2$ and a continuous utility function u_1. Prove that $\sigma_1 \leq \sigma_2$.

Exercise 1.33. Consider the mixed extension of the two-player zero-sum game "higher number wins" with $S_1 = S_2 = \mathbb{N} \setminus \{0\}$. The player who chose the higher number wins and receives a utility of 1; the losing player receives a utility of -1. In case of a draw both players get utility 0.

(a) Show that this game does not have a Nash equilibrium.

(b) Modify the rules of the game such that the player with the higher number wins only if the higher number is less than three times the lower number, and loses otherwise. Show that the modified game has a mixed Nash equilibrium.

1.6 Bibliographic Notes

The existence of a mixed Nash equilibrium in finite two-player zero-sum games was proven by von Neumann (1928). Nash (1950b) proved the existence of a Nash equilibrium for strategic games using the fixed point theorem of Kakutani (Kakutani, 1941). In the same year, Nash published a simpler proof that relies on the fixed point theorem of Brouwer (Nash, 1950a). Debreu (1952), Glicksberg (1952), and Fan (1952) independently remarked that the applications of Kakutani's fixed point theorem gives in fact gives rise to the existence of equilibria in a Euclidean games which generalize mixed extensions of finite games. For a further generalization towards joint strategy spaces that are no necessarily product spaces, see Rosen (1965).

The Hawk-Dove game in Exercise 1.28 is also known as the Chicken game. It is the symmetric version of a Hawk-Dove game studied by Smith (1982). The game in Exercise 1.29 is taken from Nau et al. (2004). The result to prove in Exercise 1.31 is due to Nash (1950a) and, in fact, applies as well to symmetric games with more than two players. The game "Higher number wins" from Exercise 1.33 shows that the finiteness of a zero-sum game is necessary for the existence of a mixed Nash equilibrium and, thus, complements von Neumann's Theorem. The game in the second part is known as Silverman's game (Heuer and Leopold-Wildburger, 1995).