Gray codes and symmetric chains

Petr Gregor[1] Sven Jäger[2] Torsten Mütze[2]
Joe Sawada[3] Kaja Wille[2]

[1]Univerzita Karlova, Praha

[2]Technische Universität Berlin

[3]University of Guelph

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Gray codes

Exhaustive listing of a class of combinatorial objects where successive objects differ by a small amount.

Examples

- All bitstrings of length d where successive bitstrings differ by a single bitflip. **[Gray 53]**
- All spanning trees of a graph where two successive spanning trees differ by exchanging a single edge. **[Cummins 66]**
- All triangulations of a regular n-gon where successive triangulations differ by a single edge-flip. **[Lucas 87]**

Applications

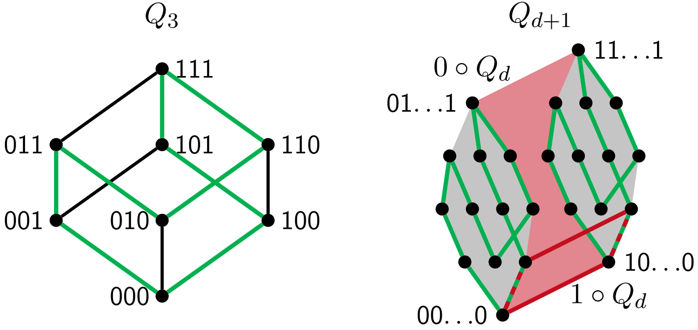
- Generate all objects in a combinatorial class quickly (small transformation in each step)
- Error correction, Boolean circuit minimization, ...

Binary reflected Gray code

Theorem [Gray 53]

For $d \in \mathbb{N}$ there is a cyclic listing of all bitstrings of length d, where two successive bistrings differ in a single bit.

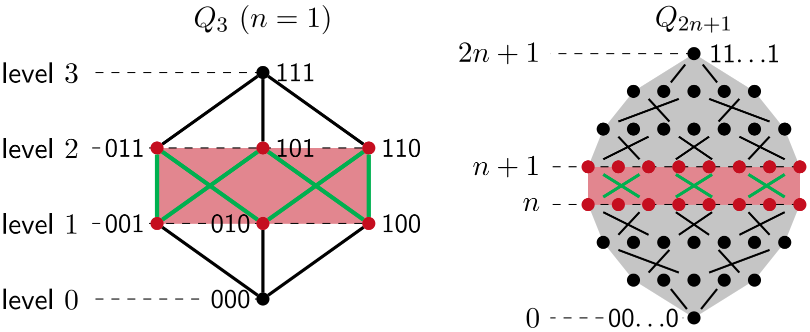
Equivalent: There is a Hamilton cycle in the d-cube.



Middle levels theorem

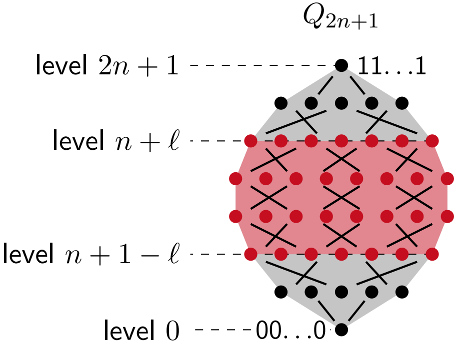
Theorem [Mütze 16]

For $n \in \mathbb{N}$ the subgraph induced by the middle two levels of the $(2n+1)$-cube has a Hamilton cycle.



Generalized middle levels conjecture

Conjecture [Savage 93, Gregor, Škrekovski 10, Shen, Williams 15] For $n \in \mathbb{N}$ and $1 \leq \ell \leq n+1$ the subgraph of the $(2n+1)$-cube induced by the middle 2ℓ levels has a Hamilton cycle.



Known results

Let $n \in \mathbb{N}$.

$\ell = n+1$  Hamilton cycle **[Gray 53]**

$\ell = n$  Hamilton cycle **[El-Hashash, Hassan 01], [Locke, Stong 03]**

$\ell = n-1$  Hamilton cycle **[Gregor, Škrekovski 10]**

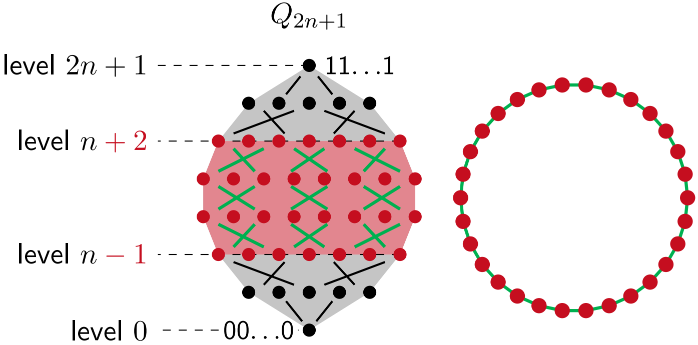
?

$\ell = 1$  Hamilton cycle **[Mütze 16]**

Our results

Theorem 1

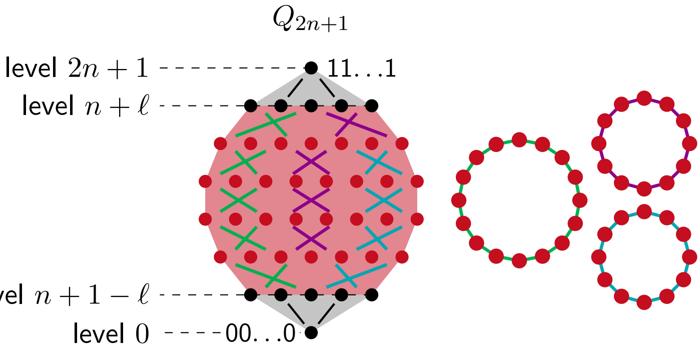
For $n \in \mathbb{N}$ the subgraph of the $(2n+1)$-cube induced by the middle **four** *levels has a Hamilton cycle.*



Our results

Theorem 2

For $n \in \mathbb{N}$ and $1 \leq \ell \leq n+1$ the subgraph of the $(2n+1)$-cube induced by the middle 2ℓ levels has a **cycle factor** *(2-factor), i.e., a spanning 2-regular subgraph.*



A cycle factor is often the first step for proving Hamiltonicity.

Known results

Let $n \in \mathbb{N}$.

$\ell = n+1$		Hamilton cycle	**[Gray 53]**
$\ell = n$		Hamilton cycle	**[El-Hashash, Hassan 01], [Locke, Stong 03]**
$\ell = n-1$		Hamilton cycle	**[Gregor, Škrekovski 10]**

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$\ell = 1$		Hamilton cycle	**[Mütze 16]**

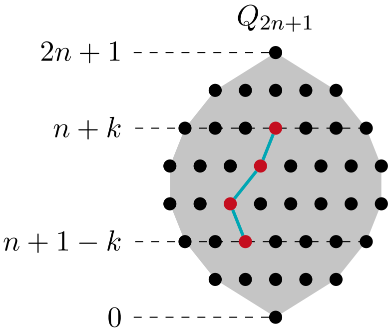
Proof of Theorem 2

Theorem 2 (Reminder)

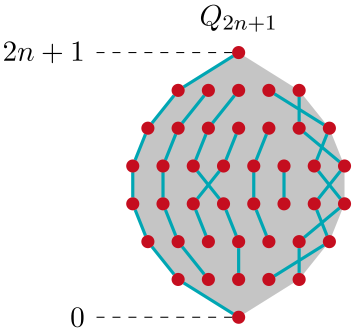
The subgraph of the $(2n+1)$-cube induced by the middle 2ℓ levels has a cycle factor.

Ingredients

Symmetric chain



Symmetric chain decomposition (SCD) **[de Bruijn et al. 51]**

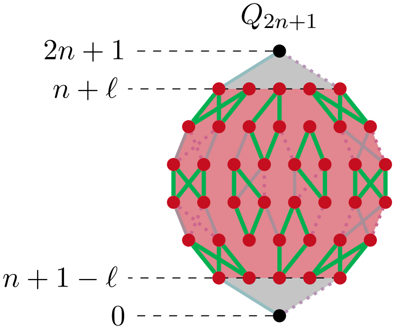


Proof of Theorem 2

Theorem 2 (Reminder)

The subgraph of the $(2n+1)$-cube induced by the middle 2ℓ levels has a cycle factor.

- Q_{2n+1} has two edge-disjoint SCDs. **[Shearer, Kleitman 79]**
- Restrict to the middle 2ℓ levels.
- Each chain has an odd number of edges.
  $\Rightarrow$ Taking every second edge from each chain yields two disjoint perfect matchings.
- Their union is a cycle factor.



Edge-disjoint SCDs in the hypercube

Theorem 3

For any $d \geq 12$ the d-cube contains four pairwise edge-disjoint SCDs.

Combining any pair of them gives six distinct cycle factors.

Known results

- Q_d has two almost orthogonal SCDs for all $d \geq 2$.
  [Shearer, Kleitman 79]
- Q_d has three pairwise almost orthogonal SCDs for all $d \geq 24$.
  [Spink 17]

Edge-disjoint SCDs in the hypercube

Theorem 3

For any $d \geq 12$ the d-cube contains four pairwise edge-disjoint SCDs.

Proof structure

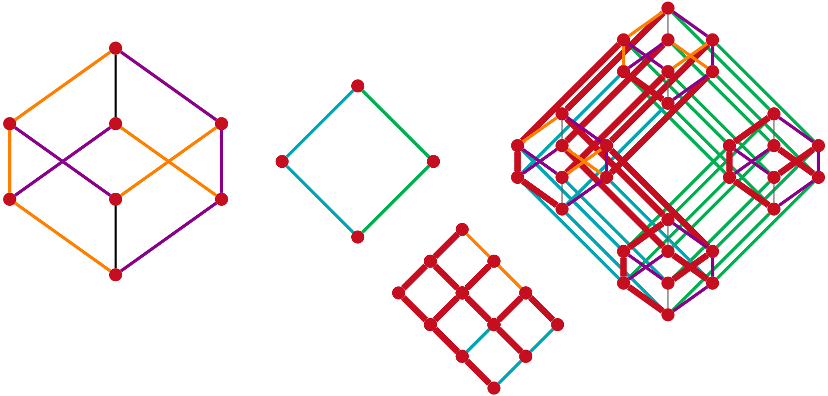
1. For *even* $d \geq 6$ there is a direct construction.
2. Q_7 contains four pairwise edge-disjoint SCDs (ad hoc construction).
3. **If Q_a and Q_b contain k pairwise edge-disjoint SCDs, then Q_{a+b} contains k pairwise edge-disjoint SCDs.**

Remark The cases $d = 6, 7$ together with Part 3 would establish the claim for all $d \geq 30$.

Product construction of SCDs

Lemma (cf. de Bruijn et al. 51, Spink 17)

Let $a, b, k \in \mathbb{N}$. If Q_a and Q_b each contain k pairwise edge-disjoint SCDs, then $Q_{a+b} \cong Q_a \,\Box\, Q_b$ contains k pairwise edge-disjoint SCDs.



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The even case

Proof is based on lexical matchings **[Kierstead, Trotter 88]** between consecutive levels.

Edge-disjoint SCDs in the hypercube

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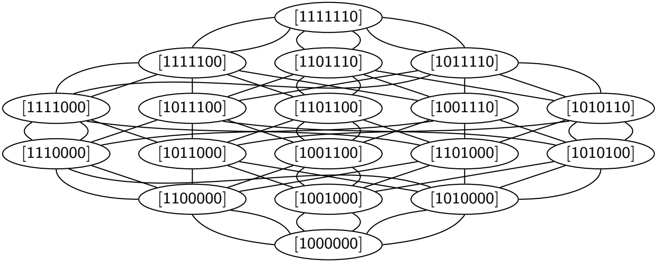
Four edge-disjoint SCDs in Q_7

Problem Brute force too slow!

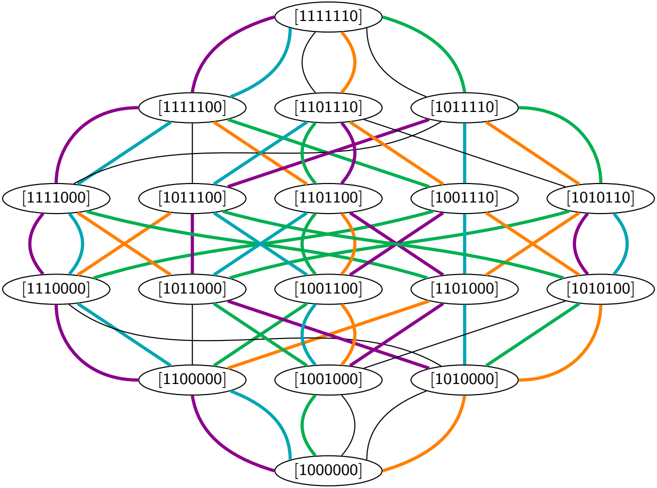
Reduce the graph

- Remove the vertices 0000000 and 1111111.
- Combine bitstrings that differ by a rotation to a single vertex representing a *necklace*.
  $\Rightarrow$ Every necklace contains 7 bitstrings (7 being prime).
- The number of edges between two necklaces $[x]$ and $[y]$ is $|N_{Q_7}(x) \cap [y]| = |[x] \cap N_{Q_7}(y)|$.



Four edge-disjoint SCDs in Q_7

Edge-disjoint SCDs in the reduced multigraph correspond to edge-disjoint SCDs in Q_7.

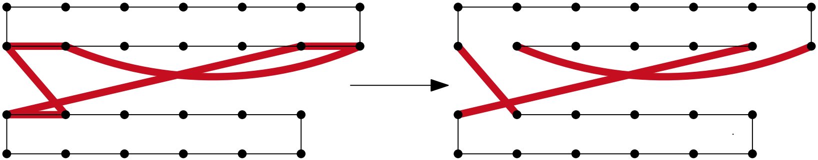


Proof Sketch of Theorem 1

Theorem 1 (Reminder)

The subgraph of the $(2n+1)$-cube induced by the middle four levels has a Hamilton cycle.

1. Build a cycle factor of the graph.
2. Join cycles by taking symmetric differences with 6-cycles.



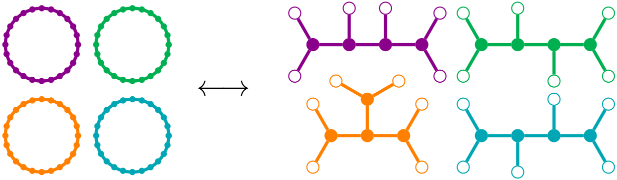
Show that all cycles can be joined to a Hamilton cycle.

Proof Sketch of Theorem 1

Theorem 1 (Reminder)

The subgraph of the $(2n+1)$-cube induced by the middle four levels has a Hamilton cycle.

Find combinatorial interpretation of cycles in the cycle factor



Characterize when two cycles can be joined.

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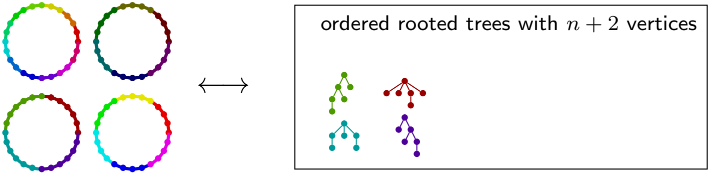

Characterize when two cycles can be joined.

- Follow cycle $\longleftrightarrow$ Do special rotation.
- Join cycle segments $\longleftrightarrow$ Do *pull* operation.

Every tree can be transformed to every other tree.

Open problems

- Analyze the new SCDs to find a combinatorial interpretation of the resulting cycle factor in order to make progress in the generalized middle levels conjecture.
- Prove or disprove that the d-cube has $\lfloor d/2 \rfloor + 1$ pairwise edge-disjoint SCDs. (cf. **Shearer, Kleitman 79**)
  - Clearly upper bound
  - True for $d \leq 7$
- Prove or disprove that almost all cubes have five pairwise edge-disjoint SCDs. (Smallest open dimension $d = 8$)

Thank you!

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