Generalizing the Kawaguchi-Kyan Bound to Stochastic Parallel Machine Scheduling

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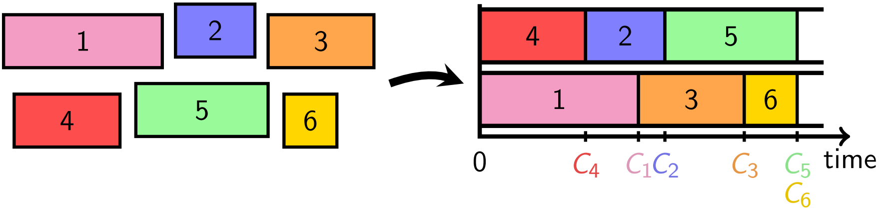
Identical Parallel Machine Scheduling ($P||\sum w_j C_j$)

Given: weights $w_j \geq 0$ and processing times $p_j \geq 0$ of jobs $j = 1, \ldots, n$ and number m of machines.

Task: Process each job nonpreemptively for p_j time units on one machine such that the total weighted completion time $\sum_{j=1}^{n} w_j C_j$ is minimized.



- Classical NP-hard problem [Garey & Johnson, problem SS13]
- Polynomial-time approximation scheme [Skutella & Woeginger 1999]

Weighted Shortest Processing Time (WSPT) Rule

WSPT rule

Whenever a machine is idle, start available job with max. ratio w_j/p_j on it.

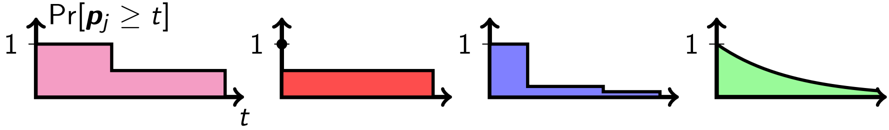
The WSPT rule is optimal for a single machine [Smith 1956] and for unit weights [Conway, Maxwell, & Miller 1967].

Theorem [Kawaguchi & Kyan 1986]

The WSPT rule is a $\frac{1}{2}(1+\sqrt{2})$*-approximation, and this bound is tight.*

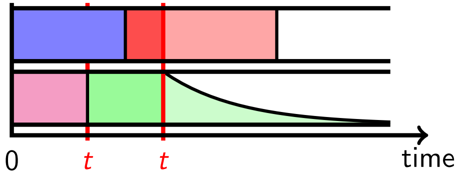
Stochastic Scheduling ($P|\boldsymbol{p}_j \sim \text{stoch}|\,\mathrm{E}[\sum w_j \boldsymbol{C}_j]$)

Given: weights $w_j \geq 0$ and distributions of independent random processing times $\boldsymbol{p}_j \geq 0$ of jobs $j = 1, \ldots, n$ and number m of machines.



Task: Find nonpreemptive scheduling policy Π minimizing the expected sum of weighted completion times.

A scheduling policy must be nonanticipative, i.e., a decision made at time t may only depend on the information known at time t.



Weighted Shortest Expected Processing Time (WSEPT) Rule

WSEPT rule

Whenever a machine is idle, start available job with largest ratio $w_j / \mathrm{E}[\boldsymbol{p}_j]$ on it.

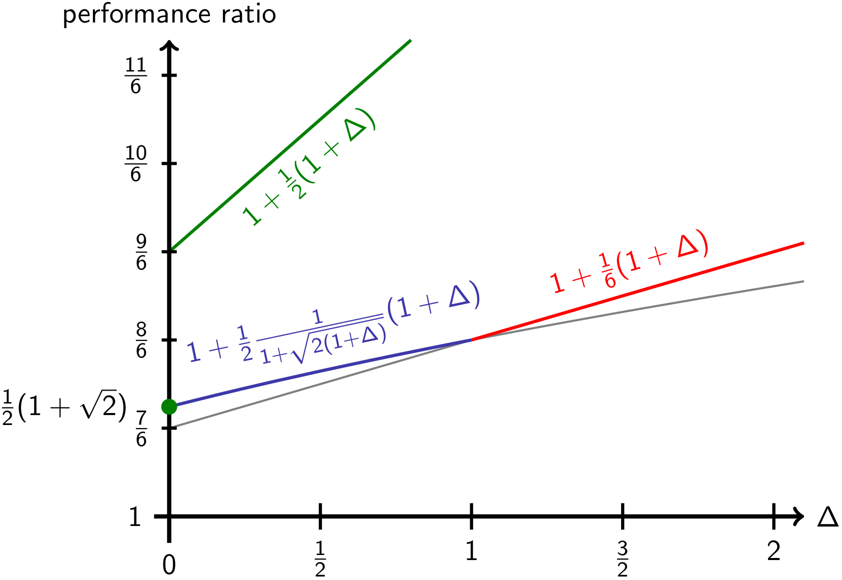
Known Results

- WSEPT has no constant performance ratio (even for unit weights). [Cheung et al. 2014; Im, Moseley, & Pruhs 2015]
- WSEPT is optimal if
  - there is only one machine [Rothkopf 1966],
  - all jobs have unit weight and processing times are pairwise stochastically comparable [Weber, Varaiya, & Walrand 1986].
- If $\frac{\mathrm{Var}[\boldsymbol{p}_j]}{\mathrm{E}[\boldsymbol{p}_j]^2} \leq \Delta$ for all j, then WSEPT has performance guarantee

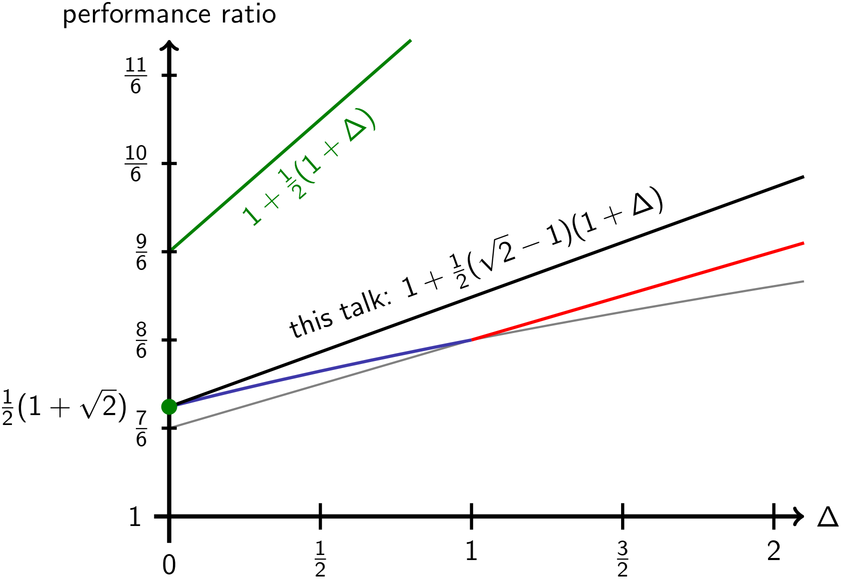
$$1 + \frac{m-1}{2m} \cdot (1+\Delta) \leq 1 + \tfrac{1}{2} \cdot (1+\Delta).$$

[Möhring, Schulz, & Uetz 1999]

Performance Guarantees



Performance Guarantees



Auxiliary Objective Function

Given: Smith ratios ρ_j and distributions of independent random processing times $\boldsymbol{p}_j \geq 0$ of jobs $j = 1, \ldots, n$ and number m of machines.

Task: Find nonpreemptive scheduling policy minimizing the expected sum of weighted completion times, where each job is weighted with its Smith ratio times its actual processing time.

- The weight of a job is a random variable $\boldsymbol{w}_j = \rho_j \boldsymbol{p}_j$.
- The Smith ratio ρ_j of a job is deterministic.

Remark

List scheduling the jobs in nonincreasing order of their Smith ratios ρ_j is a $\frac{1}{2}(1 + \sqrt{2})$-approximation for the auxiliary objective function.

Proof of WSEPT's Performance Guarantee

Claim

The WSEPT rule is a $1 + \frac{1}{2}(\sqrt{2} - 1) \cdot (1 + \Delta)$-approximation for $P|\boldsymbol{p}_j \sim \text{stoch}|\, \mathrm{E}[\sum w_j \boldsymbol{C}_j]$.

Consider auxiliary objective function with Smith ratios $\rho_j := w_j / \mathrm{E}[\boldsymbol{p}_j]$.

Then, for every policy Π:

$$\mathrm{Obj}(\Pi) = \sum_{j=1}^{n} \rho_j \, \mathrm{E}[\boldsymbol{p}_j]\, \mathrm{E}[\boldsymbol{C}_j^{\Pi}] \qquad \mathrm{Obj}'(\Pi) = \sum_{j=1}^{n} \rho_j \, \mathrm{E}[\boldsymbol{p}_j \boldsymbol{C}_j^{\Pi}]$$

original objective function value

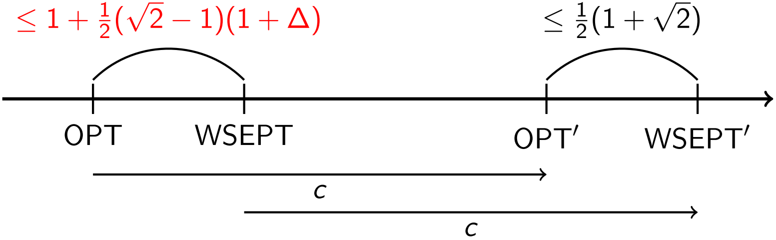
auxiliary objective function value

$$\mathrm{E}[\boldsymbol{p}_j \boldsymbol{C}_j^{\Pi}] = \mathrm{E}[\boldsymbol{p}_j(\boldsymbol{S}_j^{\Pi} + \boldsymbol{p}_j)] = \mathrm{E}[\boldsymbol{p}_j \boldsymbol{S}_j^{\Pi}] + \mathrm{E}[\boldsymbol{p}_j^2]$$
$$= \mathrm{E}[\boldsymbol{p}_j]\, \mathrm{E}[\boldsymbol{S}_j^{\Pi}] + \mathrm{E}[\boldsymbol{p}_j]^2 + \mathrm{Var}[\boldsymbol{p}_j] = \mathrm{E}[\boldsymbol{p}_j]\, \mathrm{E}[\boldsymbol{C}_j^{\Pi}] + \mathrm{Var}[\boldsymbol{p}_j].$$

nonanticipativity

Proof of WSEPT's Performance Guarantee

$$\mathrm{Obj}'(\Pi) = \mathrm{Obj}(\Pi) + \underbrace{\sum_{j=1}^{n} \rho_j \operatorname{Var}[\boldsymbol{p}_j]}_{=:c} \le \mathrm{Obj}(\Pi) + \underbrace{\sum_{j=1}^{n} \Delta w_j \operatorname{E}[\boldsymbol{p}_j]}_{c \le \Delta\,\mathrm{OPT}}.$$



$$\begin{aligned}
\mathrm{WSEPT} &= \mathrm{WSEPT}' - c \le \tfrac{1}{2}(1+\sqrt{2})\,\mathrm{OPT}' - c \\
&= \tfrac{1}{2}(1+\sqrt{2})(\mathrm{OPT} + c) - c = \mathrm{OPT} + \tfrac{1}{2}(\sqrt{2}-1)(\mathrm{OPT} + c) \\
&\le (1 + \tfrac{1}{2}(\sqrt{2}-1)(1+\Delta))\,\mathrm{OPT}
\end{aligned}$$

$c \le \Delta\,\mathrm{OPT}$

Concluding Remarks

- Considering α-points instead of completion times reduces the constant c, and thus yields the better performance guarantee.
- The derived bound is the best known performance guarantee of *any* algorithm for $P|\boldsymbol{p}_j \sim \text{stoch}|\,\mathrm{E}[\sum w_j \boldsymbol{C}_j]$.
- For $\boldsymbol{p}_j \sim \exp$, WSEPT's approximation ratio lies in $[1.243, 4/3]$ (lower bound due to Jagtenberg, Schwiegelshohn, & Uetz 2013).

  Even in this special case no better approximation is known.
- The performance guarantee can be refined for fixed numbers of machines.

Thank you!

Literature

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