Robust Combinatorial Optimization

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Outline

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1. Introduction
   - Robustness
2. Solving Robust Binary Problems
   - Weight Robust Binary Knapsack
   - Profit Robust Binary Knapsack
3. Solving Robust Non-Binary Problems
   - Extending Bertsimas/Sim
   - Profit Robust Unbounded Knapsack
   - Weight Robust Unbounded Knapsack
   - Problems with TUM Descriptions

Motivation

- Usual assumption in combinatorial optimization:
  All data known exactly in advance.
  $\longrightarrow$ In practice rarely the case.
- Include uncertainty in the models.
  $\longrightarrow$ Robust optimization, stochastic programming

A Robust Model

- Γ-scenarios:
  As part of the input, a number Γ is specified.
  Given a set of possible *events* (changes, demands, ...),
  each subset of size $\leq \Gamma$ is a scenario.
- Minmax-objective:
  Minimize the worst case cost.
- Recoverability:
  In a second stage, after the scenario is fixed, the solution might be modified (at increased cost).

Weight Robust Knapsack

Given

- a knapsack capacity $W \geq 0$,
- nominal weights $w_j \in [0, W]$,
- increased weights $w'_j \in [w_j, W]$,
- profits $p_j \geq 0$, $j \in N$,
- the maximal number of items with increased weights $\Gamma \in N$,

find $x \in \{0,1\}^n$ maximizing the profit $\sum_j p_j x_j$, such that

$$\sum_{j \in N} w_j x_j \leq W \ .$$

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find $x \in \{0,1\}^n$ maximizing the profit $\sum_j p_j x_j$, such that

$$\max_{\substack{S \subseteq N \\ |S| \leq \Gamma}} \left\{ \sum_{j \in N \setminus S} w_j x_j + \sum_{j \in S} w'_j x_j \right\} \leq W \ .$$

Profit Robust Knapsack

Given

- a knapsack capacity $W \geq 0$,
- weights $w_j \in [0, W]$,
- nominal profits $p_j \geq 0$, $j \in N$,
- decreased profits $p'_j \in [0, p_j]$,
- the maximal number of items with decreased profits $\Gamma \in N$,

find $x \in \{0,1\}^n$ such that $\sum w_j x_j \leq W$,
maximizing the *worst-case* profit

$$\min_{\substack{S \subseteq N \\ |S| \leq \Gamma}} \left\{ \sum_{j \in N \setminus S} p_j x_j + \sum_{j \in S} p'_j x_j \right\} .$$

Outline

A Dynamic Program (1/2)

Preprocessing: Sort items such that

$$\Delta w_1 \geq \Delta w_2 \geq \ldots \geq \Delta w_n, \quad \text{with } \Delta w_j := w'_j - w_j$$

State space:

$$[Y, P, m] \in \mathcal{S}_j \mathrel{\widehat{=}} \text{solution using items from } \{1, 2, \ldots, j\},$$
$$\text{where} \quad Y = \text{weight},$$
$$P = \text{profit},$$
$$m = \text{number of packed items.}$$

A Dynamic Program (2/2)

Constructing the new states:

For $[Y, P, m] \in \mathcal{S}_{j-1}$,

- Always
$$[Y, P, m] \in \mathcal{S}_j$$
- If $m \geq \Gamma$ and $Y + w_j \leq W$
$$[Y + w_j, P + p_j, m + 1] \in \mathcal{S}_j$$
- If $m < \Gamma$ and $Y + w'_j \leq W$
$$[Y + w'_j, P + p_j, m + 1] \in \mathcal{S}_j$$

Return the solution with maximal profit in $\mathcal{S}_n$.

Results

By elimination of dominated states $|\mathcal{S}_j| \leq (W+1)(n+1)$.

Theorem

The Dynamic Program for the Weight Robust Knapsack Problem can be implemented with running time $\mathcal{O}(n^2 W)$.

Applying a result from Woeginger (1999) we can turn the DP into an FPTAS:

Theorem

There is an FPTAS for the Weight Robust Knapsack Problem.

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Bertsimas and Sim, 2003

Cost Robust Binary Minmax Problems with Γ-scenarios:

$$\min_{x \in X} \left\{ c^\intercal x + \max_{\substack{S \subseteq N \\ |S| \leq \Gamma}} \sum_{j \in S} d_j x_j \right\}$$

$\Downarrow$ duality

$$\min_{\substack{x \in X \\ \vartheta \geq 0}} \left\{ c^\intercal x + \Gamma\vartheta + \sum_{j \in N} \max\{d_j x_j - \vartheta, 0\} \right\}$$

$\Downarrow X \subseteq \{0,1\}^n$

$$\min_{\substack{x \in X \\ \vartheta \geq 0}} \left\{ c^\intercal x + \Gamma\vartheta + \sum_{j \in N} \max\{d_j - \vartheta, 0\} \cdot x_j \right\}$$

$\Downarrow$

Bertsimas and Sim, 2003

$$\min_{x\in X}\left\{c^\top x+\max_{\substack{S\subseteq N\\|S|\le\Gamma}}\sum_{j\in S}d_jx_j\right\}$$

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$\Downarrow$

Bertsimas and Sim, 2003

$$\min_{x\in X}\left\{c^{\mathsf{T}}x + \max_{\substack{S\subseteq N\\ |S|\leq\Gamma}} \sum_{j\in S} d_j x_j\right\}$$

$\Downarrow$ duality

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$\Downarrow$

$(n+1)$ possible values for optimal ϑ, for fixed ϑ linear in x.

Bertsimas and Sim, 2003

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$(n+1)$ possible values for optimal ϑ, for fixed ϑ linear in x.

Results

Applying Bertsimas/Sim:

Corollary

There is an FPTAS for the Profit Robust Knapsack Problem.

Applying Bertsimas/Sim on the Weight Robust Knapsack:

Corollary

There is an FPTAS for the General Robust Knapsack Problem with uncertain profits and uncertain weights.

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Outline

Main Result

Theorem

Consider the problem $\min_{x \in X} c^T x$ with $c \geq 0$ and $X \subseteq \mathbb{N}_0^n$. If

- *there is a ρ-approximation algorithm for $\min_x \sum_j \tilde{c}_j(x_j)$ for piecewise linear cost functions with a single bend $\tilde{c}_j$,*
- *bounds u_j on the variables can be computed in polynomial time,*
- *(i) $d_j u_j$ is polynomial for all j or*
  (ii) $\log u_j$ is polynomial for all j

then there is a

(i) ρ-approximation algorithm
(ii) $(2\rho + \varepsilon)$-approximation algorithm, for any $\varepsilon > 0$,

for the cost robust problem with Γ-scenarios, respectively.

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Proof of the Main Result (1/2)

$$\min_{x \in X} \left\{ c^{\mathsf{T}} x + \max_{\substack{S \subseteq N \\ |S| \leq \Gamma}} \sum_{j \in S} d_j x_j \right\}$$

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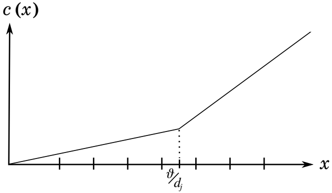
Piecewise linear cost functions with a single bend (for fixed ϑ).



Proof of the Main Result (2/2)

For fixed ϑ, there is a ρ-approximation algorithm.

$$\vartheta > \max_j d_j u_j \quad \Rightarrow \quad \max\{d_j x_j - \vartheta, 0\} = 0$$
$$\Rightarrow \quad \vartheta^* \le \max_j d_j u_j \ .$$

1. $d_j u_j$ is polynomial for all j:
   $\Rightarrow$ polynomially many possible values for ϑ^*
   $\Rightarrow$ enumeration maintains approximation factor.
2. $\log u_j$ is polynomial for all j:
   $\Rightarrow$ try $\vartheta = 0, 1, (1+\varepsilon), (1+\varepsilon)^2, \ldots$
   $\Rightarrow$ ϑ^* missed by a factor of at most $(1+\varepsilon)$
   $\Rightarrow$ additional factor of $(2+\varepsilon)$. □

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Outline

The Unbounded Knapsack Problem

Given

- a knapsack capacity W
- n types of items with weights w_j and profits p_j

find a vector $x \in \mathbb{N}_0^n$ such that

$$\sum_{j \in N} w_j x_j \leq W \qquad \text{and} \qquad \sum_{j \in N} p_j x_j \longrightarrow \max$$

Goal: Solve the profit robust version.

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Profit Robust Unbounded Knapsack

Main Result also holds for maximization problems if the relative cost decrease is bounded:

$$\exists\, \alpha > 0 : \qquad \frac{(c_j - d_j)}{c_j} \geq \alpha \quad \forall\, j \in N\,.$$

Apply the result to the Unbounded Knapsack Problem:

- Bounds on the variables:

$$x_j \leq \left\lfloor \frac{W}{w_j} \right\rfloor =: u_j \quad \Rightarrow \quad \log u_j \text{ polynomial}$$

- Solve the problem with piecewise linear cost functions: Consider the *Concave* Unbounded Knapsack Problem

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Concave Unbounded Knapsack Problem

ConcUKP:
An Unbounded Knapsack Problem with concave profit functions instead of constant profit coefficients.

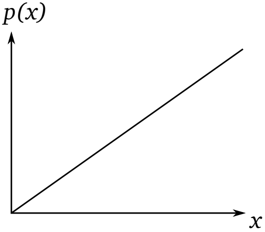

Figure: The profit functions of UKP and ConcUKP

Concave Unbounded Knapsack Problem

ConcUKP:

An Unbounded Knapsack Problem with concave profit functions instead of constant profit coefficients.

Lemma

There is an FPTAS for ConcUKP.

Concave Unbounded Knapsack Problem

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Lemma

There is an FPTAS for ConcUKP.

Proof.

Modify the standard DP (with states $[Y, P]$):
For every type consider to pack $0, 1, (1+\varepsilon), (1+\varepsilon)^2, \dots$ items of this type. This causes only $(1+\varepsilon)$ loss $(*)$.
The DP can be transformed into an FPTAS via Woeginger. □

Proof of $(*)$

x^* optimal, $\overline{x}$ the corresponding DP solution:

$$\overline{x}_j = (1+\varepsilon)^{\lfloor \log_{(1+\varepsilon)} x_j^* \rfloor} \quad \Rightarrow \quad \overline{x}_j \geq \frac{1}{1+\varepsilon} x_j^* \quad \forall\, j \in N$$

Then

$$p_j(\overline{x}_j) \geq p_j\left(\frac{1}{1+\varepsilon} x_j^*\right) \qquad (p_j \text{ increasing})$$

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$$\geq \frac{1}{1+\varepsilon} p_j(x_j^*) + \left(1 - \frac{1}{1+\varepsilon}\right) p_j(0) \qquad (p_j \text{ concave})$$

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Then

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$$\Rightarrow \quad \sum_{j \in N} p_j(\overline{x}_j) \geq \frac{1}{1+\varepsilon} \cdot \text{OPT}$$

Results

Corollary

For any $\varepsilon > 0$, there is a $(2 + \varepsilon)$-approximation algorithm for the Profit Robust Unbounded Knapsack Problem with bounded relative profit decrease.

Outline

Results

Modifying the DP from above in the same way as for the binary case and applying Woeginger's result again leads to:

Theorem

There is an FPTAS for the Weight Robust Unbounded Knapsack Problem.

Theorem

For any $\varepsilon > 0$, there is a $(2 + \varepsilon)$-approximation algorithm for the General Robust Unbounded Knapsack Problem with bounded relative profit decrease.

Outline

Problems with TUM Descriptions

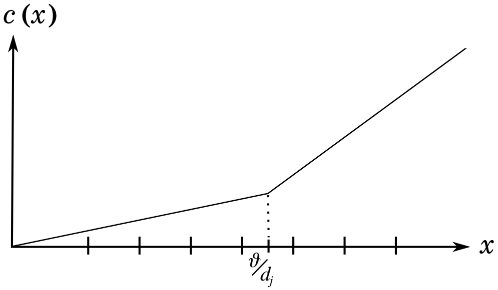
Lemma

If there is a totally unimodular description of the solution space, i.e.

$$\text{conv}(X) = \{x \in \mathbb{R}^n : Ax \leq b\}, \quad \text{with } A \text{ TUM}, b \in \mathbb{Z}^n$$

then the problem with piecewise linear cost functions can be solved exactly in polynomial time.

Splitting the Variables



Split each variable into three, two of them bounded, with different cost coefficients, to get a modified problem.

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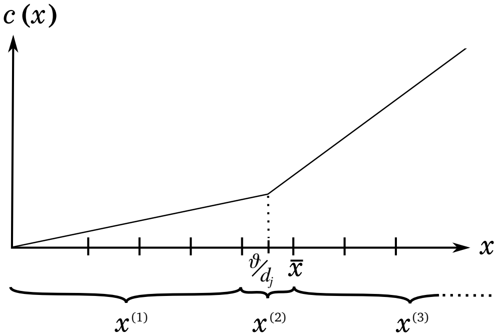

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The modified problem

Variable	Upper Bound	Cost Coefficient
$x_j^{(1)}$	$\overline{x}_j - 1$	$c_j^{(1)} := c_j$
$x_j^{(2)}$	1	$c_j^{(2)} := c_j + c'_j \overline{x}_j - \alpha_j$
$x_j^{(3)}$	-	$c_j^{(3)} := c_j + c'_j$

Since $c_j^{(1)} \leq c_j^{(2)} \leq c_j^{(3)}$,

$$\min_{x \in X} \sum_{j \in N} \widetilde{c}_j(x_j) = \begin{cases} \min & c^{(1)}x^{(1)} + c^{(2)}x^{(2)} + c^{(3)}x^{(3)} \\ \text{s.t.} & x^{(1)} + x^{(2)} + x^{(3)} \in X \\ & x^{(1)} \leq \overline{x} - \mathbb{1} \\ & x^{(2)} \leq \mathbb{1} \, . \end{cases}$$

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A TUM description of the modified problem

$$\begin{cases} x^{(1)} + x^{(2)} + x^{(3)} \in \text{conv}(X) \\ x^{(1)} \leq \overline{x} - \mathbb{1} \\ x^{(2)} \leq \mathbb{1} \end{cases}$$

$$\iff \underbrace{\begin{bmatrix} A & A & A \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}}_{=:A'} \underbrace{\begin{bmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{bmatrix}}_{=:x'} \leq \underbrace{\begin{bmatrix} b \\ \overline{x} - \mathbb{1} \\ \mathbb{1} \end{bmatrix}}_{=:b'} \qquad (*)$$

A' TUM, $b' \in \mathbb{Z}^{3n}$
$\Rightarrow$ optimal solution of $(*)$ integral
$\Rightarrow$ problem exactly solvable. $\square$

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Conclusion

- To solve cost robust non-binary combinatorial optimization problems, one has to study the deterministic problems with piecewise linear cost functions with a single bend.
- The results for the Unbounded Knapsack Problem and for problems with a totally unimodular description illustrate that this is possible.
- Uncertainty in the weights adds a linear factor to the complexity of the Binary and the Unbounded Knapsack Problem, approximability is preserved.

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Thank you for your attention!