Multicriteria Optimization and Compromise Solutions

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(Joint work with Christina Büsing and Jannik Matuschke)

MDS Colloquium – 07/02/2011

Multicriteria Combinatorial Optimization

$$\max f(x)$$
$$\text{s.t. } x \in \mathcal{X}$$

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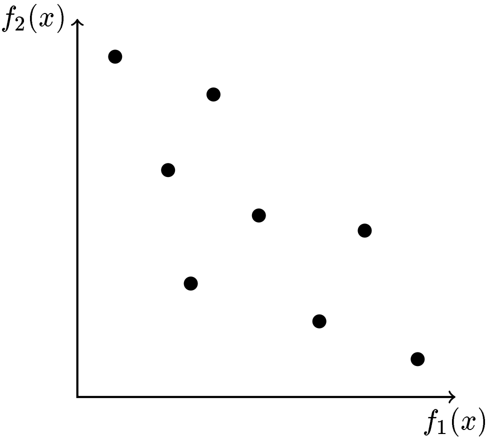
How to optimize this?

Practitioner's state-of-the-art

Scalarization: Maximize (weighted) sum of all objectives.



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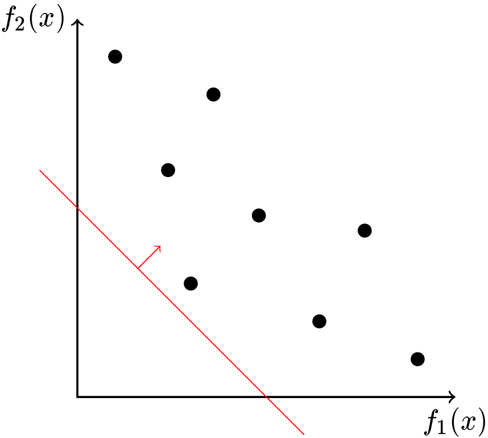

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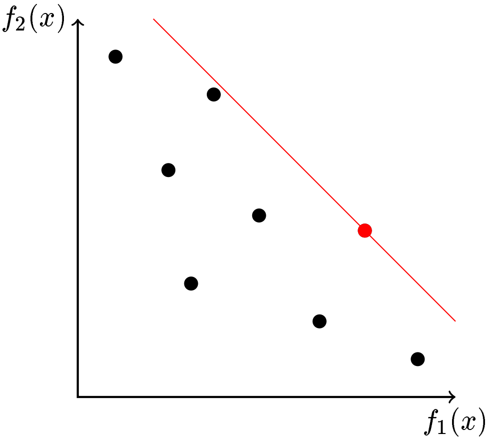

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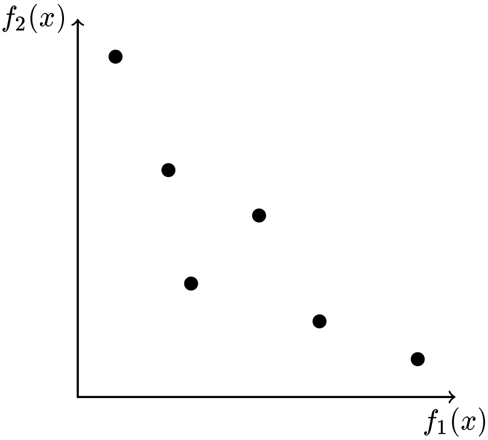

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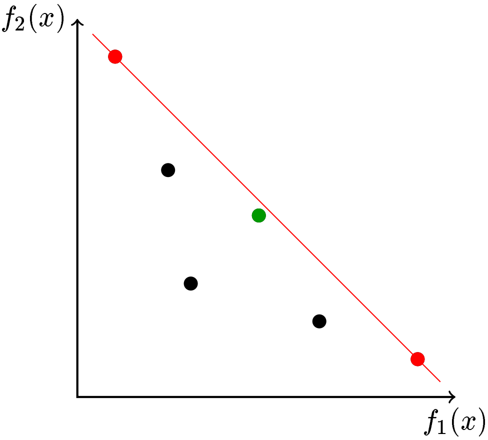

Mathematician's answer

Pareto optimal solutions:
No objective can be improved without worsening another.

Definition (Pareto optimal solution)

A solution $x \in \mathcal{X}$ is *Pareto optimal*, if

$$\nexists\, x' \in \mathcal{X} : f(x') \geq f(x) \quad \text{and} \quad f_i(x') > f_i(x) \text{ for some } i$$

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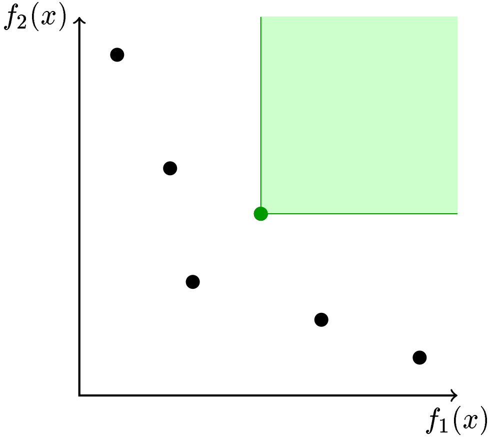

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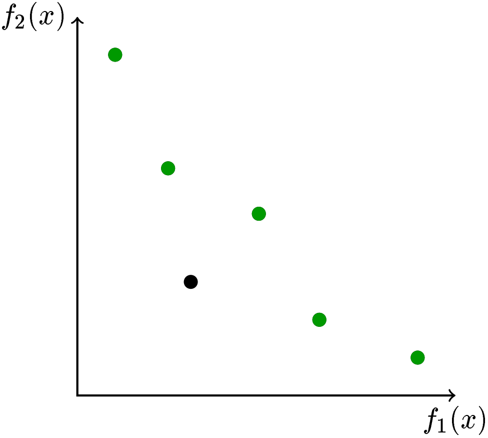

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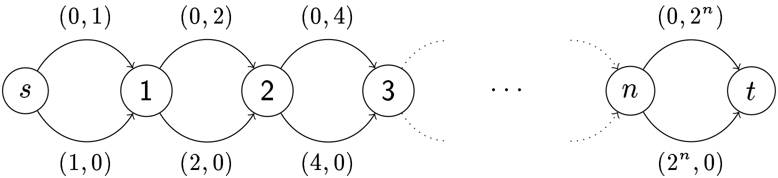
Problem:

Number of Pareto optimal solutions might be exponential.

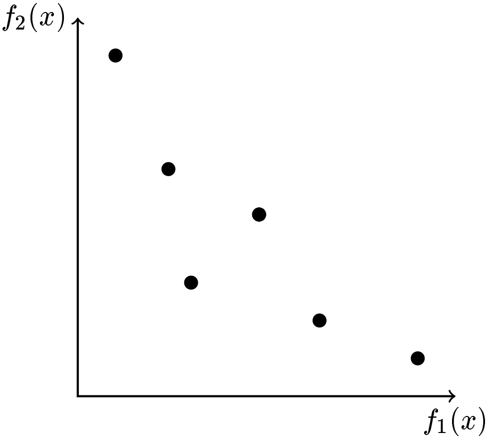


All 2^{n+1} (s,t)-paths are Pareto optimal.

Our Answer

Compromise Solutions:

Minimize distance to ideal point (w.r.t. some metric).



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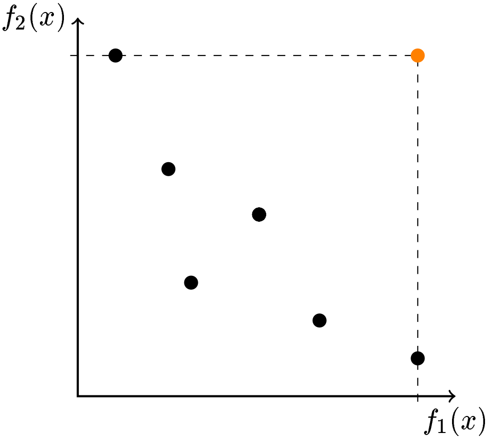

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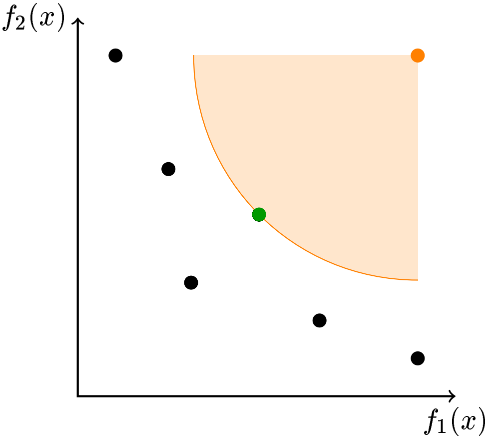

Notation:

$$f(x) := (f_1(x), f_2(x), \dots, f_k(x)) = \text{cost vector of a solution } x$$

$$\mathcal{Y} := \{f(x) : x \in \mathcal{X}\}$$

$$\begin{aligned}\mathcal{N} :=& \{y \in \mathcal{Y} : \nexists y' \in \mathcal{Y} \text{ s.t. } y' \geq y \text{ and } y'_i > y_i \text{ for some } i\} \\ =& \{\text{cost vectors of Pareto optimal solutions}\} \\ =& \{\textit{non-dominated} \text{ cost vectors}\}\end{aligned}$$

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Definition (Ideal Point)

Given a multicriteria optimization problem

$$\max_{x \in \mathcal{X}} (f_1(x), \ldots, f_k(x)),$$

the *ideal point* $y^* = (y_1^*, \ldots, y_k^*)$ is defined by

$$y_i^* = \max_{x \in \mathcal{X}} f_i(x) = \max_{y \in \mathcal{Y}} y_i \qquad \forall\, i.$$

Definition (Compromise Solution)

Given a multicriteria optimization problem

$$\max_{x \in \mathcal{X}} (f_1(x), \dots, f_k(x))$$

with the ideal point $y^* \in \mathbb{Q}^k$, the *Compromise Solution* w.r.t. the norm $\|\cdot\|$ on $\mathbb{Q}^k$ is

$$x^{\mathsf{CS}} := \operatorname*{argmin}_{x \in \mathcal{X}} \|y^* - f(x)\|$$

Remark: We often identify solutions and cost vectors.
For example, we will speak of the Compromise Solution

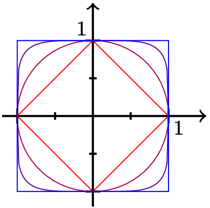
$$y^{\mathsf{CS}} = \min_{y \in \mathcal{Y}} \|y^* - y\|.$$

The norms we consider:

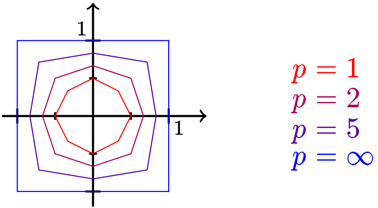
$$\|y\|_p := \left(\sum_{i=1}^{k} y_i^p\right)^{1/p}, \quad p \in [1, \infty) \qquad (L^p\text{-Norm})$$

$$\|y\|_\infty := \max_{i=1,\dots,k} y_i \qquad (\text{Maximum } (L^\infty\text{-})\text{Norm})$$

$$|||y|||_p := \|y\|_\infty + \frac{1}{p}\|y\|_1, \quad p \in [1, \infty] \qquad (\textit{Cornered } p\text{-Norm})$$



L^p-Norm



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Weighted version: For any norm and $\lambda \in \mathbb{Q}^k, \lambda \geq 0, \lambda \neq 0$:

$$\|y\|^\lambda = \|(\lambda_1 y_1, \lambda_2 y_2, \dots, \lambda_k y_k)\|,$$

e.g. $$|||y|||_p^\lambda = \max_i\{\lambda_i y_i\} + \frac{1}{p}\lambda^\intercal y.$$

Notation: For a family of norms $\|\cdot\|_p, p \in [1, \infty]$, define

$$\mathsf{CS}(\lambda, p) := \{\text{Compromise Solution w.r.t. } \|\cdot\|_p^\lambda\}$$
$$\mathsf{CS}(p) := \{\mathsf{CS}(\lambda, p) : \lambda \in \mathbb{Q}^k, \lambda \geq 0, \lambda \neq 0\}$$

Assumption: $\mathcal{Y}$ closed.

Gearhardt 1979: Compromise solutions w.r.t. to the L^p-Norm and the Cornered Norm have the following properties:

- Pareto optimality: $\mathsf{CS}(p) \subseteq \mathcal{N}$ for $p < \infty$.
- $\bigcup_{p\in[1,\infty)} \mathsf{CS}(p)$ is dense in $\mathcal{N}$.
- If $\mathcal{N}$ bounded:

$$\sup_{y\in\mathcal{N}} \mathrm{dist}_\infty(y, \mathsf{CS}(p)) \xrightarrow[p\to\infty]{} 0. \qquad (\mathrm{dist}_\infty(y, A) = \inf_{y'\in A} \|y - y'\|_\infty)$$

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$\mathcal{Y}$ discrete $\Rightarrow \mathcal{N} = \mathsf{CS}(p)$ for p big enough. *How big?*
From now on, consider the Cornered Norm $\| \cdot \|_p$.

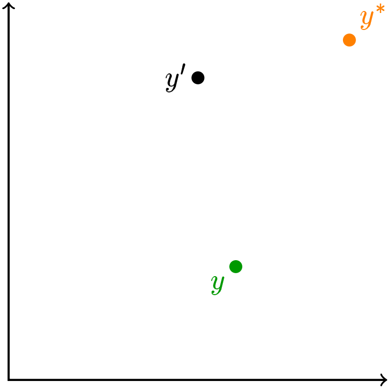
Assumption: $\mathcal{Y} \subset \mathbb{N}_0^k$, and $\exists$ polynomial π s.t. for any instance I with encoding length $|I|$, $y_i \leq 2^{\pi(|I|)}$ for all $y \in \mathcal{Y}, i = 1, \ldots, k$.

Lemma (Büsing,G.,Matuschke)

Let $M := 2^{\pi(|I|)}$. For $p > kM$, it holds that $\mathcal{N} = \mathsf{CS}(p)$.

Remark: Since $\log(kM) = \log(k) + \pi(|I|)$, this shows that p can be chosen such that it has polynomial encoding length.

Proof: Let y^* be the ideal point, $y, y' \in \mathcal{N}$ arbitrary.



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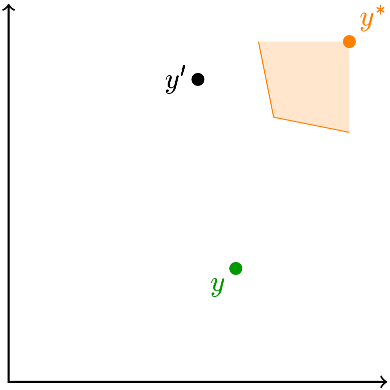

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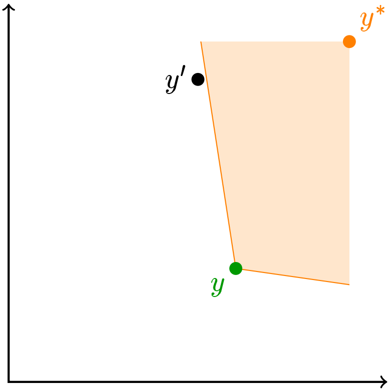

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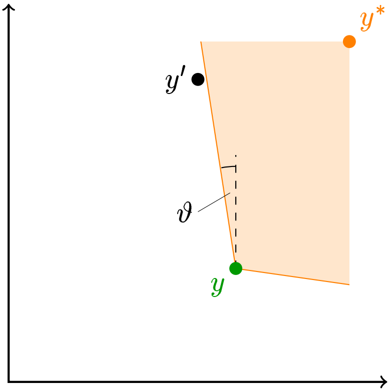

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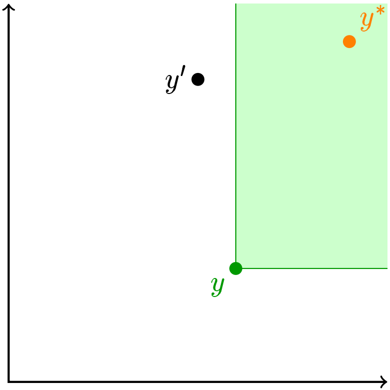

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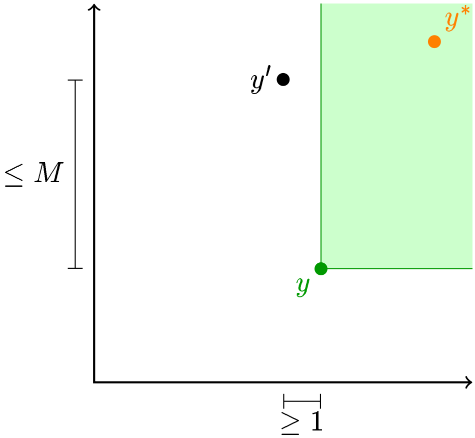
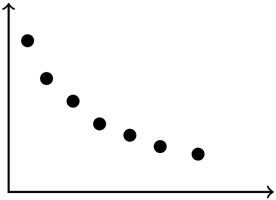


Another way to cope with exponential size of $\mathcal{N}$:

Definition (ε-approximate Pareto set)

Let $\mathcal{X}_P$ be the Pareto set of a given instance, and let $\varepsilon > 0$. $\mathcal{X}_{\varepsilon P} \subseteq \mathcal{X}$ is an *ε-approximate Pareto set* if for all $x \in \mathcal{X}_P$ there is $x' \in \mathcal{X}_{\varepsilon P}$ such that

$$f_i(x) \leq (1+\varepsilon) f_i(x') \quad \forall i = 1, \ldots, k$$



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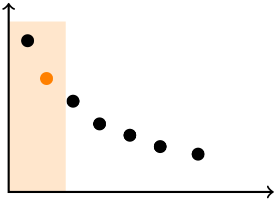

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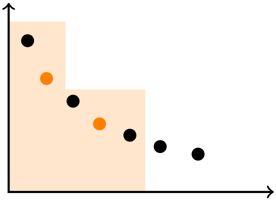

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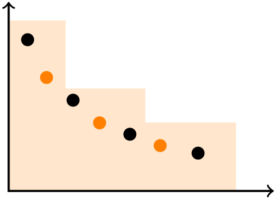

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Theorem (Papadimitriou&Yannakakis,2000)

There always exists an ε-approximate Pareto set with size polynomial in $|I|$ and $1/\varepsilon$.

Idea: Partition criterion space into hyperrectangles

$$[(1+\varepsilon)^{\ell_1}, (1+\varepsilon)^{\ell_1+1}] \times \cdots \times [(1+\varepsilon)^{\ell_k}, (1+\varepsilon)^{\ell_k+1}],$$

$$\text{where } \ell_i = 0, 1, \dots, \log_{(1+\varepsilon)} M$$

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Idea: Partition criterion space into hyperrectangles and choose one point from each (if there is one).

How to find an ε-approximate Pareto set?

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Definition (GAP problem)

The GAP problem is the following:
Given a vektor $y \in \mathbb{Q}^k$ and $\varepsilon > 0$,
either return a solution $x \in \mathcal{X}$ with $f_i(x) \geq y_i \ \forall i$,
or answer that there is no $x \in \mathcal{X}$ with $f_i(x) \geq (1+\varepsilon) y_i \ \forall i$.

Theorem (Papadimitriou&Yannakakis,2000)

There is an algorithm for constructing an ε-approximate Pareto set if and only if the GAP problem is tractable.

Idea: Partition criterion space into hyperrectangles
and solve GAP problem for each lower left corner.

Connection to Compromise Solutions:

Theorem (Büsing,G.,Matuschke)

For $p > \dfrac{kM^2}{\varepsilon}$ *, an FPTAS for* CS(p) *solves the* $\textsc{Gap}$ *problem.*

Remark: This p still has polynomial encoding length.

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Remark: This p still has polynomial encoding length.
Proof: Let $y \in \mathbb{Q}^k, \varepsilon > 0$ be given. W.l.o.g.

$$y < y^* \qquad \text{(otherwise answer to } \textsc{Gap} \text{ is } \textsc{No}\text{)},$$
$$\lfloor y \rfloor \neq 0 \qquad \text{(otherwise } \textsc{Gap} = \textsc{Feasibility}\text{)},$$
$$\varepsilon < \frac{1}{y_i} \; \forall i \; .$$

First consider $y \in \mathbb{N}^k$, $y_i \neq 0 \; \forall i$ for simplicity.

For all i, set $\lambda_i := \frac{1}{y_i^* - y_i}$.
($\rightsquigarrow y =$ "lower left corner" of ball around y^*.)

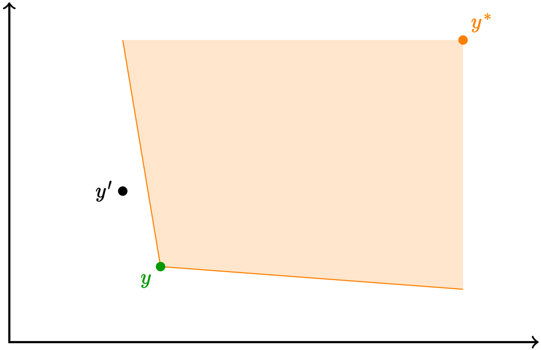
Let y' be returned by the FPTAS for CS(p, λ) for some ε'.

Positive Case:

$$\|y^* - y'\|_p^\lambda \leq \|y^* - y\|_p^\lambda \quad \overset{p > kM}{\Longrightarrow} \quad y' \geq y$$

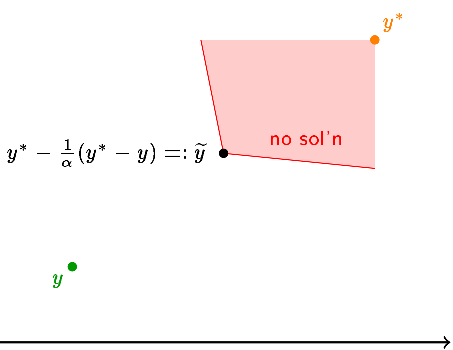
$$\rightsquigarrow \quad \text{RETURN } y'.$$

Negative Case: $\|y^* - y'\|_p^\lambda > \|y^* - y\|_p^\lambda$



Negative Case: $\|y^* - y'\|_p^\lambda > \|y^* - y\|_p^\lambda$

$$\Rightarrow \quad \nexists y'' \in \mathcal{Y} : \|y^* - y''\|_p^\lambda \leq \frac{1}{\alpha}\|y^* - y\|_p^\lambda \qquad (\alpha = 1 + \varepsilon').$$



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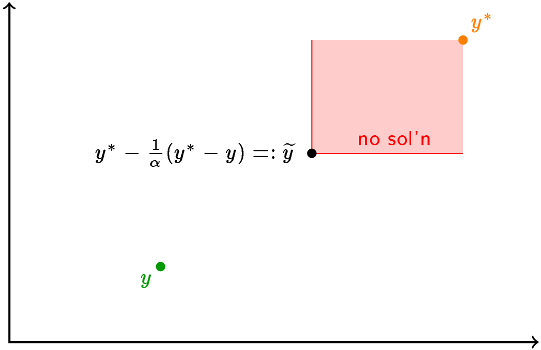

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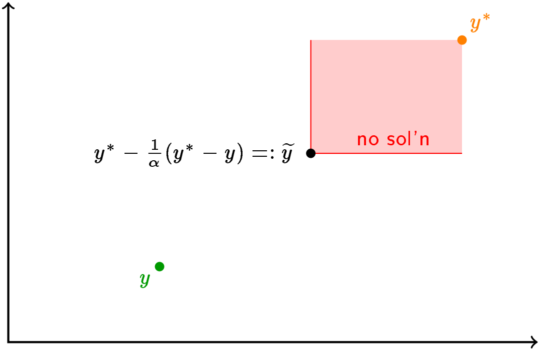

$$\Rightarrow \quad \text{Need } \widetilde{y} \leq (1 + \varepsilon)y.$$

Choosing ε':

$$\widetilde{y}_i = y_i^* - \frac{1}{\alpha}(y_i^* - y_i) \leq (1+\varepsilon)y_i \quad \Leftrightarrow \quad \alpha \leq \frac{y_i^* - y_i}{y_i^* - y_i - \varepsilon y_i}$$

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$$\frac{y_i^* - y_i - \varepsilon y_i}{y_i^* - y_i} \leq 1 - \frac{\varepsilon y_i}{y_i^* - y_i} \leq 1 - \frac{\varepsilon}{M} \qquad (y_i \geq 1)$$

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$$\frac{y_i^* - y_i - \varepsilon y_i}{y_i^* - y_i} \leq 1 - \frac{\varepsilon y_i}{y_i^* - y_i} \leq 1 - \frac{\varepsilon}{M} \qquad (y_i \geq 1)$$

$$\Rightarrow \quad \alpha = \frac{1}{1 - \varepsilon/M} \quad \text{or} \quad \varepsilon' = 1 - \alpha = 1 - \frac{1}{1 - \varepsilon/M}$$

$\Rightarrow$ For this ε' and $y \in \mathbb{N}^k$, FPTAS for CS(p, λ) answers Gap.

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If $y' \geq y$, Return y'.

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If $y' \geq y$, RETURN y'.

Otherwise, choosing

$$\alpha = \frac{1 - \frac{1}{\beta}}{1 - \frac{\varepsilon}{M}}, \qquad \beta > \frac{M}{\varepsilon}, \qquad p \geq \beta k M > \frac{kM^2}{\varepsilon}$$

guarantees that there is no $y'' \in \mathcal{Y} : y'' \geq (1+\varepsilon) y$.

A Trivial Approximation

Scalarization: $\max_{y \in \mathcal{Y}} \lambda^\mathsf{T} y$ yields a $\frac{k+\frac{k}{p}}{1+\frac{k}{p}}$-approximation to CS$(p, \lambda)$.

(Remark: $\frac{k+\frac{k}{p}}{1+\frac{k}{p}} \leq \min\{k, p+1\}$)

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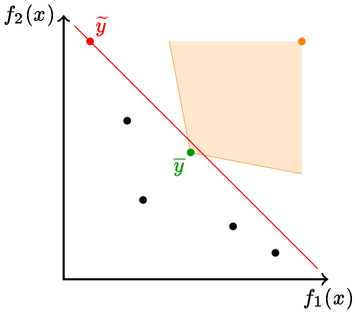
Proof:

$\widetilde{y} = \max_{y \in \mathcal{Y}} \lambda^{\mathsf{T}} y$
$\overline{y} = \mathrm{CS}(p, \lambda)$

Worst Case:
$\exists C: \ \lambda_i(y_i^* - \overline{y}_i) = C \quad \forall i$



$$
\begin{aligned}
\|y^* - \widetilde{y}\|_p^\lambda &= \max_i \{\lambda_i(y_i^* - \widetilde{y}_i)\} + \frac{1}{p}\lambda^\intercal (y^* - \widetilde{y}) \\
&\leq \left(1 + \frac{1}{p}\right) \lambda^\intercal (y^* - \widetilde{y}) \\
&\leq \left(1 + \frac{1}{p}\right) \lambda^\intercal (y^* - \overline{y}) && \text{since } \lambda^\intercal \widetilde{y} \geq \lambda^\intercal \overline{y} \\
&= \left(1 + \frac{1}{p}\right) \cdot kC
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&= C\left(1 + \frac{k}{p}\right) \\
&\Rightarrow \quad \frac{\|y^* - \widetilde{y}\|_p^\lambda}{\|y^* - \overline{y}\|_p^\lambda} \leq \frac{k + \frac{k}{p}}{1 + \frac{k}{p}}
\end{aligned}$$

Theorem (Aissi et al., '06/'07)

For a constant number of criteria, CS$(p = \infty)$ for Shortest Path and Min Spanning Tree both admit an FPTAS.

Remark:

Results originally from robust optimization,
Min-Max-Regret $\approx$ CS(∞).

Possibly extendable to $p < \infty$.

Problem: Algorithms based on approximating the set of all non-dominated regret-vectors $\approx$ Approximation of Pareto set. This is what we want to *avoid*!

Open Questions

- Do the results from Aissi et al. extend to $p < \infty$?
- Can we approximate CS(p) for simple problems without approximating the Pareto set? (Needs to be better than Scalarization.)
- Can we prove hardness/inapproximability for some problems?
- Do the general results also hold for L^p-Norm?

Thank you for your attention.

Questions are welcome!