Optimization over Integers with Robustness in Cost and Few Constraints

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(Joint work with Sebastian Stiller and Claudio Telha)

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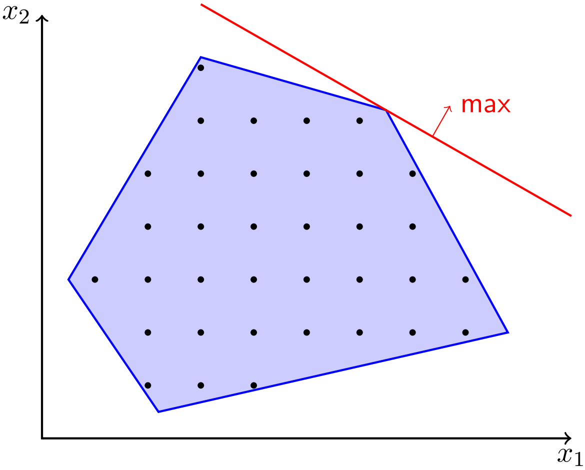
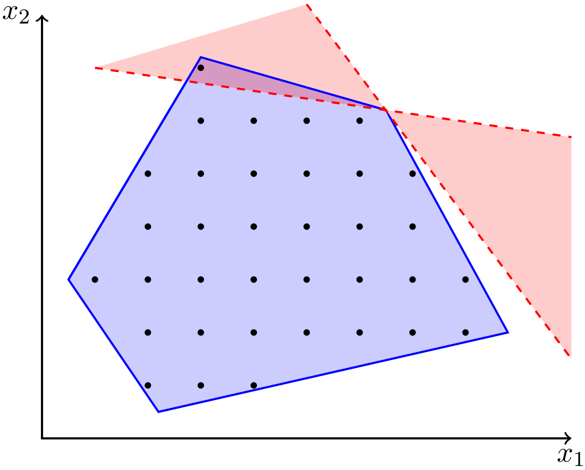
Classical Optimization over Integers.



e.g. totally unimodular IPs, Unbounded Knapsack Problem, IPs with two variables per inequality, ...

Classical Optimization over Integers. But what if. . . ?



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Cost Robust Counterpart

Given

- set of feasible solutions $X \subseteq \mathbb{Z}^n$
- cost vector $c \in \mathbb{Q}^n$

find $x \in X$ minimizing the cost

$$c^{\mathsf{T}} x$$

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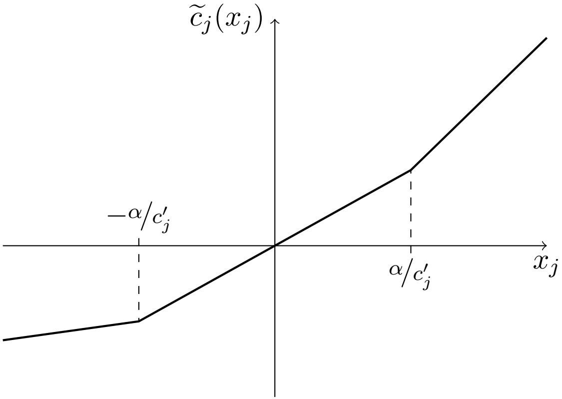
$$c^\intercal x + \max_{\substack{S \subseteq [n] \\ |S| \leq \Gamma}} \sum_{i \in S} |d_i x_i|$$

This defines the (d, Γ)-CRC of $P = (c, X)$.

To solve CRC of P: Solve the (α, c')-MMin of P.

$$\min_{x \in X} \sum_{j=1}^{n} \widetilde{c}_j(x_j), \qquad \widetilde{c}_j(x_j) = c_j + \max\{c'_j x_j - \alpha, 0\} + \max\{-c'_j x_j - \alpha, 0\}$$



Theorem

Let $X \subseteq \mathbb{Z}^n, c \in \mathbb{Q}^n, d \in \mathbb{N}^n, \Gamma \in [n]$*. If*

- *there is a* ρ*-approximation algorithm for the* (α, d)*-MMin of* P *for all* $\alpha \geq 0$
- *bounds* u_j *on the absolute value of* x_j *can be computed in polynomial time,*

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Proof (Sketch):

$$\min_{x \in X} \left\{ c^\mathsf{T} x + \max_{\substack{S \subseteq [n] \\ |S| \leq \Gamma}} \sum_{j \in S} |d_j x_j| \right\}$$

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$\Downarrow$ duality

$$\min_{\substack{x \in X \\ \vartheta \geq 0}} \left\{ c^\mathsf{T} x + \Gamma \vartheta + \sum_{j=1}^{n} \big(\max\{d_j x_j - \vartheta, 0\} + \max\{-d_j x_j - \vartheta, 0\} \big) \right\}$$

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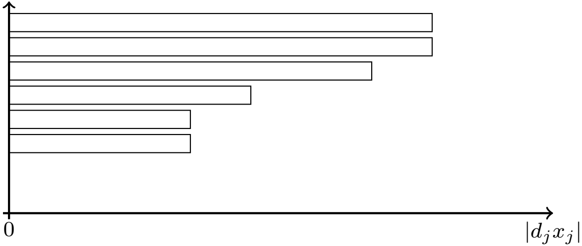

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$$\min_{x\in X}\left\{c^\mathsf{T}x+\max_{\substack{S\subseteq[n]\\|S|\leq\Gamma}}\sum_{j\in S}|d_jx_j|\right\}$$

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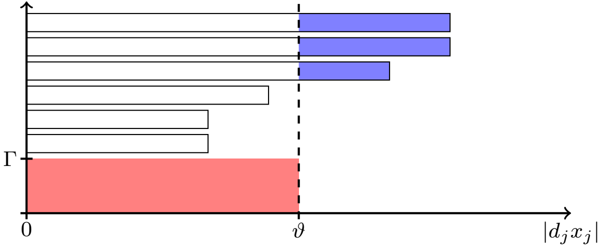

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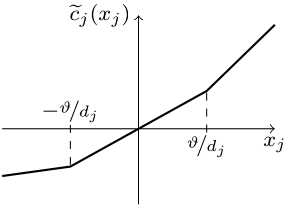
$$= \min_{\vartheta\geq 0}\left\{\Gamma\vartheta + \min_{x\in X}\left\{\sum_{j=1}^{n}\widetilde{c}_j(x_j)\right\}\right\}$$



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- $\vartheta^* \leq \max_j\{d_j u_j\}$

Finding the optimal ϑ

$$\min_{\substack{x \in X \\ \vartheta \geq 0}} \left\{ c^\top x + \Gamma\vartheta + \sum_{j=1}^{n} \Big(\max\{\underbrace{d_j x_j}_{\in \mathbb{Z}} - \vartheta, 0\} + \max\{\underbrace{-d_j x_j}_{\in \mathbb{Z}} - \vartheta, 0\} \Big) \right\}$$

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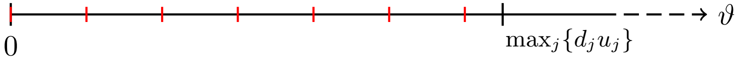

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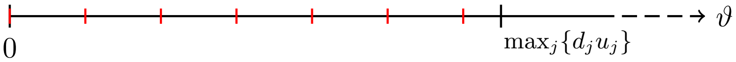

$\Rightarrow$ only $\max_j\{d_j u_j\} + 1$ possible values for ϑ^*,
can enumerate in pseudopolynomial time. □

Theorem (Extension 1)

Let $X \subseteq \mathbb{Z}^n, c \in \mathbb{Q}^n, d \in \mathbb{N}^n, \Gamma \in [n]$. If

- *there is a ρ-approximation algorithm for the (α, d)-MMin of P for all $\alpha \geq 0$*
- *bounds u_j on the absolute value of x_j can be computed in polynomial time,*

then there is a pseudopolynomial ρ-approximation algorithm for the (d, Γ)-CRC of $P = (c, X)$.

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Proof (Idea): Binary search for ϑ^*.

Theorem (Extensions 2 and 3)

Let $X \subseteq \mathbb{Z}^n, c \in \mathbb{Q}^n, d \in \mathbb{N}^n, \Gamma \in [n]$. *If*

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If $X \subseteq \mathbb{N}^n$ and $c \geq 0$, then for all $\varepsilon > 0$ there is a polynomial time $\rho(1+\varepsilon)$-approximation algorithm for the (d, Γ)-CRC of P for minimization problems.

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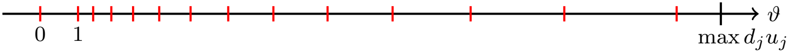
If $X \subseteq \mathbb{N}^n$ *and* $c \geq 0$, *and* $\frac{d_j}{c_j} \leq \beta < 1$,

then there is a polynomial time 2ρ*-approximation algorithm for the* (d, Γ)*-CRC of* P *for maximization problems.*

Proof (Idea):

Try all $\vartheta \in \{0, 1, 1+\varepsilon, (1+\varepsilon)^2, (1+\varepsilon)^3, \ldots, \max_j(d_j u_j)\}$.



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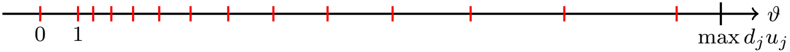

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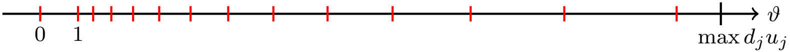

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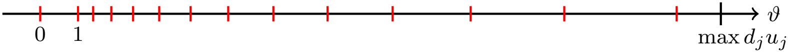

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Maximization with $\frac{d_j}{c_j} \leq \beta$, setting $\varepsilon := \frac{1-\beta}{2\beta}$
$\rightsquigarrow$ missing optimum by at most 2ρ. □

Theorem (Extension 2)

Let $X \subseteq \mathbb{Z}^n, c \in \mathbb{Q}^n, d \in \mathbb{N}^n, \Gamma \in [n]$. *If*

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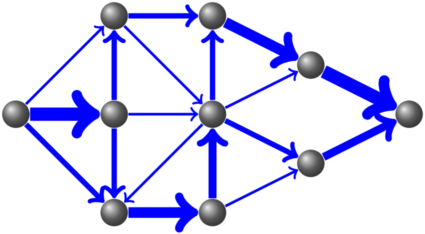
Totally Unimodular Problems

If the minimization problem $P = (c, X)$ is given by a *bounded totally unimodular description*, i.e.

$$\text{conv}(X) = \{x \in \mathbb{R}^n : Ax \leq b, x \leq u\},$$

$$A \text{ TUM}, b \in \mathbb{Z}^m, u \in \mathbb{Z}^n,$$



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then

- the (α, c')-MMin of P has a bounded TUM description,
- the optimal values of the (α, c')-MMin of P are convex in α.

$\Rightarrow$ The (d, Γ)-CRC of P is efficiently solvable.

IPs with two variables per inequality

Bounded IP2:

$$\min_{x\in\mathbb{Z}^n}\{c^\intercal x : \quad a_i^\intercal x \le b_i \text{ for } i = 1,\dots,m, \quad \ell \le x \le u\}$$

with $b \in \mathbb{Z}^m, \ell, u \in \mathbb{Z}^n, c \in \mathbb{Q}^n$,
$a_i \in \mathbb{Q}^n$ with only two non-zero entries.

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- Pseudopolynomial exact algorithm for MMin of bounded *monotone* IP2s
  $\Rightarrow$ Pseudopolynomial exact algorithm for the CRC.

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- Pseudopolynomial exact algorithm
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$\Rightarrow$ For any $\varepsilon > 0$, there is a $(2+\varepsilon)$-approximation algorithm for the (d, Γ)-CRC of UKP, if $\frac{d_j}{c_j} \leq \beta < 1$.

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Unbounded Knapsack Problem:

cost robust:	polynomial $(2+\varepsilon)$-approximation, exact in $\mathcal{O}(\max_j(d_j)n^2W^2)$
weight robust:	exact in $\mathcal{O}(\max_j(\Delta w_j)n^2W^2)$
cost & weight robust:	exact in $\mathcal{O}(\max_j(d_j)\max(\Delta w_j)n^2W^3)$

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Thank you for your attention.


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