Compromise Solutions

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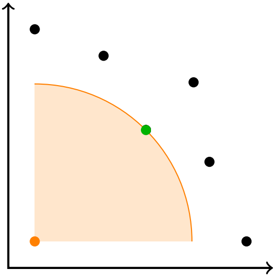
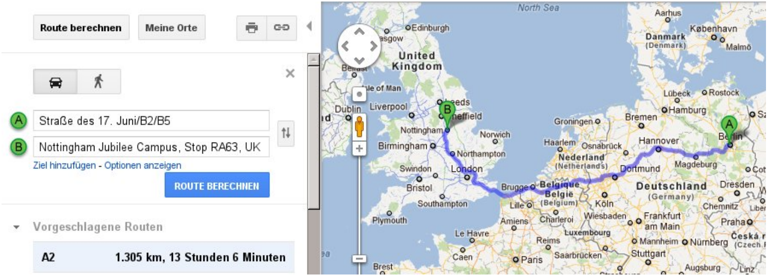
(Joint work with Christina Büsing, Jannik Matuschke and Sebastian Stiller)



SCOR 2012

Multicriteria Optimization



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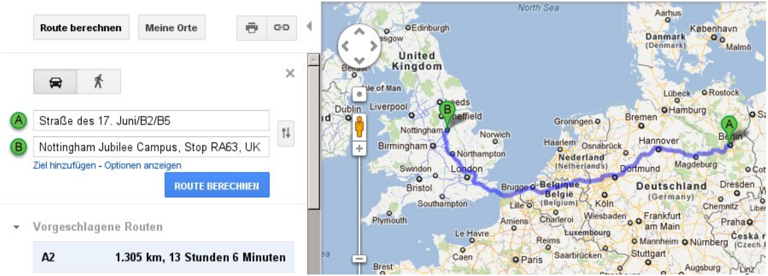

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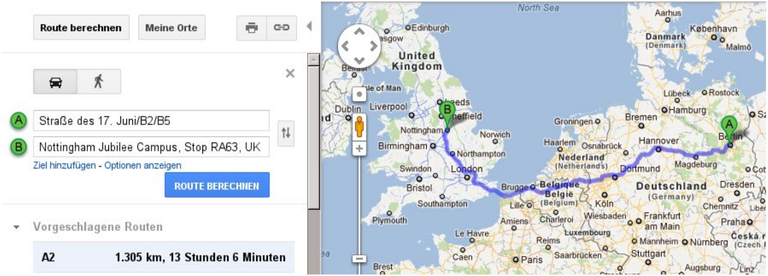

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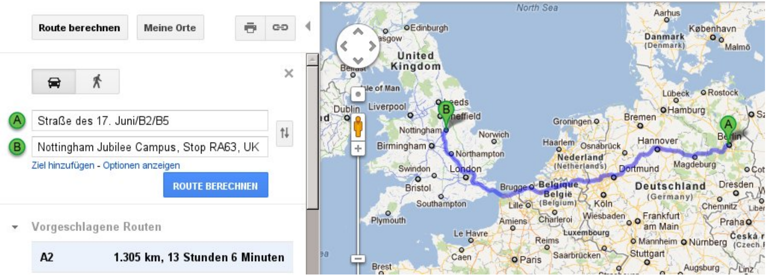

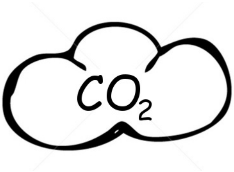

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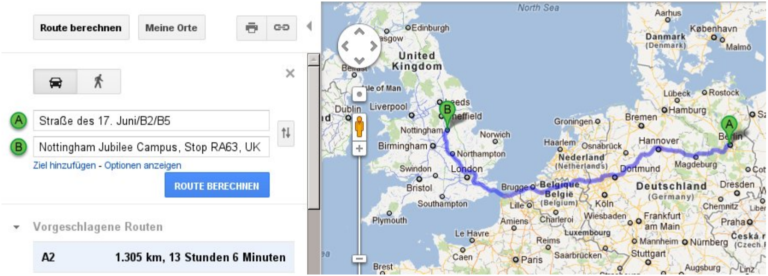
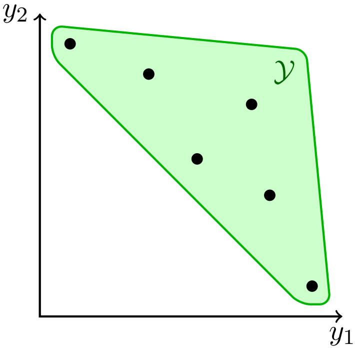


$$\min\{y : y \in \mathcal{Y}\} \qquad \text{where} \quad \mathcal{Y} \subseteq \mathbb{Z}^k$$

Pareto Optimality

Definition

A solution $y \in \mathcal{Y}$ of $\min_{y \in \mathcal{Y}} y$ is *Pareto optimal* if there is no $y' \in \mathcal{Y} \setminus \{y\}$ with $y' \leq y$.



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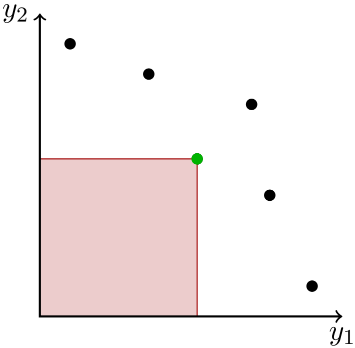

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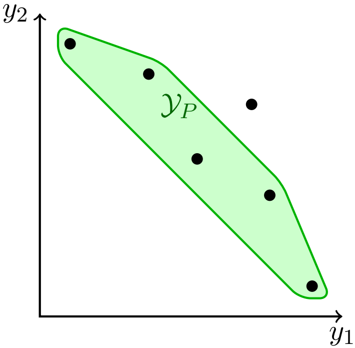
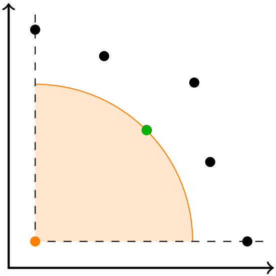


Compromise Solutions

Motivation:

- identify a single, Pareto optimal, balanced solution
- *reference point methods*:
  part of many state-of-the-art MCDM tools,
  little theoretical background



1 Introduction

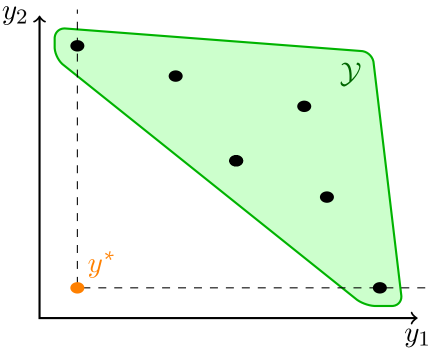
2 Definitions and Notations

3 Approximation

Definition (Ideal Point)

Given a multicriteria optimization problem $\min_{y \in \mathcal{Y}} y$, the *ideal point* $y^* = (y_1^*, \ldots, y_k^*)$ is defined by

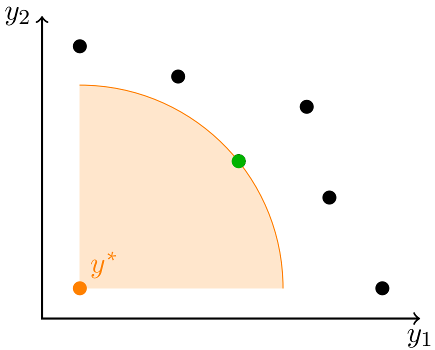
$$y_i^* = \min_{y \in \mathcal{Y}} y_i \qquad \forall\ i.$$



Definition (Compromise Solution, Yu 1973)

Given a multicriteria optimization problem $\min_{y \in \mathcal{Y}} y$ with the ideal point $y^* \in \mathbb{Q}^k$, the *compromise solution* w.r.t. the norm $\|\cdot\|$ on $\mathbb{R}^k$ is

$$y^{\text{cs}} = \min_{y \in \mathcal{Y}} \|y - y^*\| .$$



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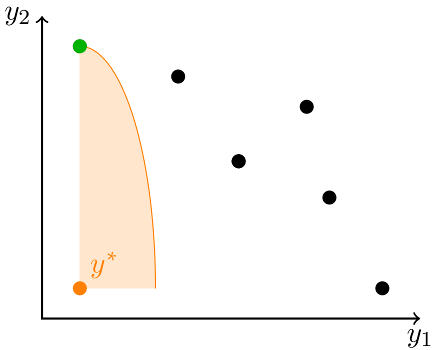
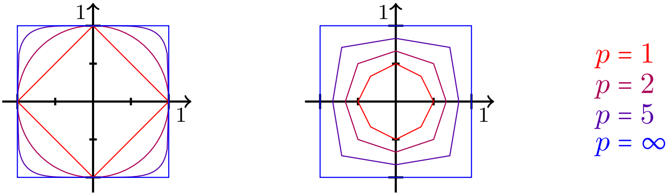


The norms we consider:

$$\|y\|_p := \left(\sum_{i=1}^{k} y_i^p\right)^{1/p}, \quad p \in [1, \infty) \qquad (\ell^p\text{-Norm})$$

$$\|y\|_\infty := \max_{i=1,\ldots,k} y_i \qquad (\text{Maximum } (\ell^\infty\text{-})\text{Norm})$$

$$\langle\langle y \rangle\rangle_p := \|y\|_\infty + \frac{1}{p}\|y\|_1, \quad p \in [1, \infty] \qquad (\textit{Cornered } p\text{-Norm})$$



ℓ^p-Norm

Cornered p-Norm

Degree of balancing controlled by adjusting p.

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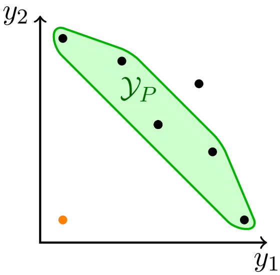
Weighted version: For any norm and $\lambda \in \mathbb{Q}^k, \lambda \geq 0, \lambda \neq 0$:

$$\|y\|^\lambda = \|(\lambda_1 y_1, \lambda_2 y_2, \dots, \lambda_k y_k)\|.$$

Known Properties

Gearhardt 1979:

- for $p < \infty$ all compromise solutions are Pareto optimal
- all Pareto optimal solution are a compromise solution, for p big enough

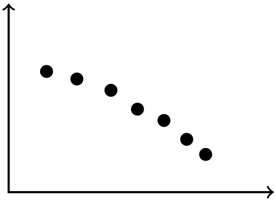


Approximate Pareto sets

Definition (α-approximate Pareto set)

Let $\mathcal{Y}_P$ be the Pareto set of $\min_{y \in \mathcal{Y}} y$, and let $\alpha \geq 1$.
$\mathcal{Y}_\alpha \subseteq \mathcal{Y}$ is an α*-approximate Pareto set* if for all $y \in \mathcal{Y}_P$ there is $y' \in \mathcal{Y}_\alpha$ such that

$$y'_i \leq \alpha y_i \quad \forall i = 1, \dots, k$$



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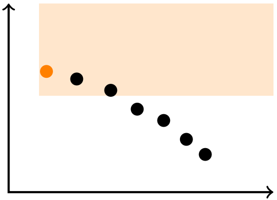

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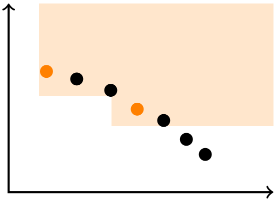

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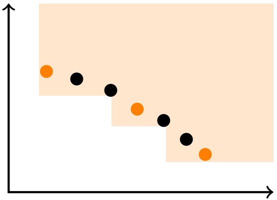

How to find approximate Pareto sets

Theorem (Papadimitriou&Yannakakis,2000)

$\textsc{Gap}(y, \alpha)$ *tractable for all* $y \in \mathbb{Q}^k$
$\Rightarrow$ α^2*-approximation for the Pareto set.*

α*-approximation for the Pareto set*
$\Rightarrow$ $\textsc{Gap}(y, \alpha)$ *tractable for all* $y \in \mathbb{Q}^k$*.*

$\textsc{Gap}(y, \alpha)$: Given $y \in \mathbb{Q}^k$ and $\alpha \geq 1$.

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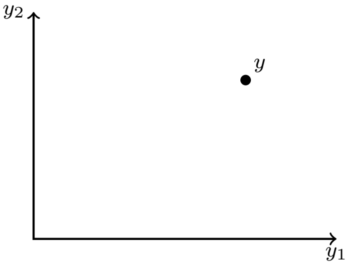

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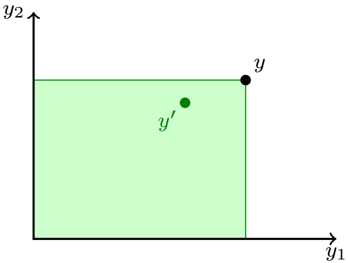

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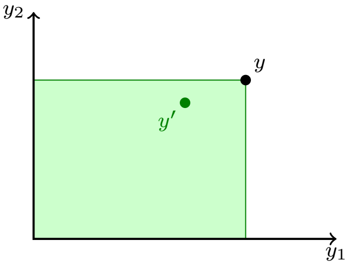

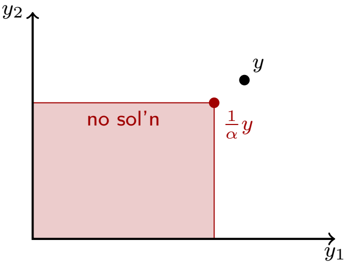

Approximate Pareto sets $\Leftrightarrow$ approximate CS

Approximate Pareto sets ⇔ approximate CS

Relate objective value to size of the vectors:

$$\min_{y \in \mathcal{Y}} \underbrace{\|y - y^*\| + \|y^*\|}_{r(y)}$$

We call $r(y)$ the *relative distance*.

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α-approximation of the Pareto set
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α-approximation for $\min_{y \in \mathcal{Y}} r(y)$
$\Rightarrow$ *$\textsc{Gap}(y, \beta)$ tractable for all $y \in \mathbb{Q}^k$, $\beta \in \Theta(\alpha)$.*
$\Rightarrow$ *β^2-approximation for the Pareto set.*

Summary

approximate Pareto set

$\Downarrow \quad \Uparrow$

approximate compromise solutions

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Thank you for your attention.


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