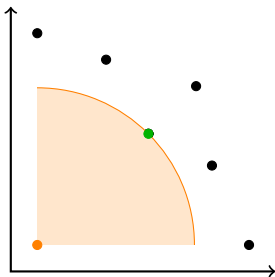


Compromise Solutions

Kai-Simon Goetzmann, TU Berlin

(Joint work with Christina Büsing, Jannik Matuschke and Sebastian Stiller)



SCOR 2012

Multicriteria Optimization

The screenshot displays a route planning interface with a map of Europe. A blue route is highlighted, starting in the UK, passing through London, and ending in Berlin. The interface includes a search bar with two stops, a 'ROUTE BERECHNEN' button, and a list of suggested routes.

Route Details:

- Stop A:** Straße des 17. Juni/B2/B5
- Stop B:** Nottingham Jubilee Campus, Stop RA63, UK

Vorgeschlagene Routen:

Route ID	Distance and Duration
A2	1.305 km, 13 Stunden 6 Minuten

Multicriteria Optimization

The screenshot shows a route planning interface. On the left, there are input fields for starting point (A) and destination (B). The starting point is 'Straße des 17. Juni/B2/B5' and the destination is 'Nottingham Jubilee Campus, Stop RA63, UK'. Below these fields is a 'ROUTE BERECHNEN' button. Underneath, a section titled 'Vorgeschlagene Routen' (Suggested Routes) shows a route labeled 'A2' with a distance of '1.305 km, 13 Stunden 6 Minuten'. On the right, a map displays the route in blue, starting from the UK and ending in Berlin, Germany. The map includes labels for various countries and cities across Europe.



Multicriteria Optimization

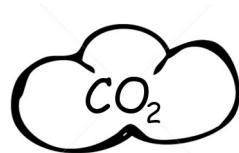
The screenshot shows a route planning interface with the following elements:

- Buttons:** "Route berechnen", "Meine Orte", "Print", "Refresh", "Map navigation controls", "Car icon", "Walking icon", "Close (X)", "Up/Down arrows", "Route Berechnen (blue button)".
- Input Fields:**
 - Point A: "Straße des 17. Juni/B2/B5" (Berlin, Germany)
 - Point B: "Nottingham Jubilee Campus, Stop RA63, UK" (Nottingham, UK)
- Route Suggestion:**
 - Route A2: 1.305 km, 13 Stunden 6 Minuten
- Map:** A map of Europe showing a blue route starting in Nottingham, UK, passing through London, Paris, and Berlin.



Multicriteria Optimization

The screenshot shows a route planning interface. On the left, there are buttons for "Route berechnen" and "Meine Orte". Below, there are input fields for two stops: "A Straße des 17. Juni/B2/B5" and "B Nottingham Jubilee Campus, Stop RA63, UK". A "ROUTE BERECHNEN" button is visible. Below the input fields, a section titled "Vorgeschlagene Routen" shows a route labeled "A2" with a distance of "1.305 km, 13 Stunden 6 Minuten". On the right, a map shows a blue route starting in London, passing through Birmingham and Nottingham, and ending in Berlin. The map includes labels for various European cities and countries like the United Kingdom, Belgium, Netherlands, and Deutschland.



Multicriteria Optimization

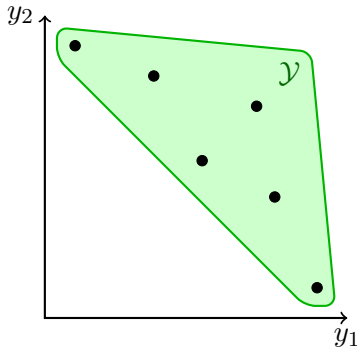
The screenshot shows a route planning interface. On the left, there are input fields for start and end points. The start point is 'Straße des 17. Juni/B2/B5' (marked with a green circle 'A') and the end point is 'Nottingham Jubilee Campus, Stop RA63, UK' (marked with a green circle 'B'). Below the input fields is a blue button labeled 'ROUTE BERECHNEN'. Underneath, a section titled 'Vorgeschlagene Routen' (Suggested Routes) shows a single route 'A2' with a distance of '1.305 km, 13 Stunden 6 Minuten'. On the right, a map of Europe shows a blue route starting in London, passing through the Netherlands and Germany, and ending in Berlin (marked with a green circle 'A').

$$\min\{y : y \in \mathcal{Y}\} \quad \text{where} \quad \mathcal{Y} \subseteq \mathbb{Z}^k$$

Pareto Optimality

Definition

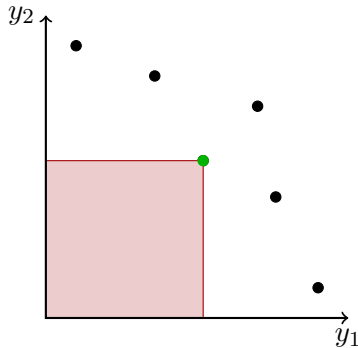
A solution $y \in \mathcal{Y}$ of $\min_{y \in \mathcal{Y}} y$ is *Pareto optimal* if there is no $y' \in \mathcal{Y} \setminus \{y\}$ with $y' \leq y$.



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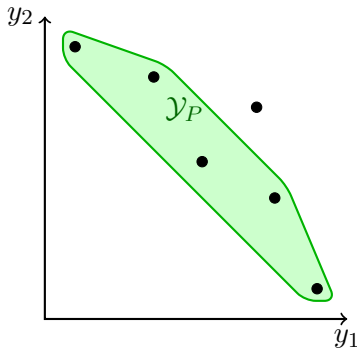
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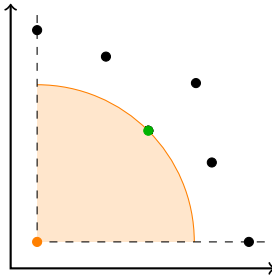
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Compromise Solutions

Motivation:

- ▶ identify a single, Pareto optimal, balanced solution
- ▶ *reference point methods*:
part of many state-of-the-art MCDM tools,
little theoretical background



1 Introduction

2 Definitions and Notations

3 Approximation

1 Introduction

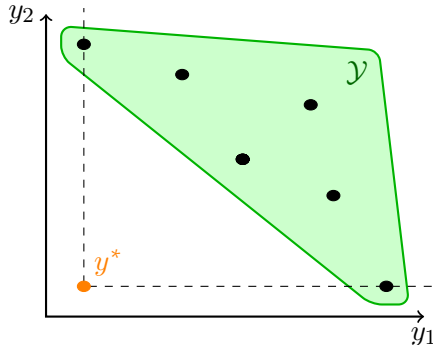
2 Definitions and Notations

3 Approximation

Definition (Ideal Point)

Given a multicriteria optimization problem $\min_{y \in \mathcal{Y}} y$, the *ideal point* $y^* = (y_1^*, \dots, y_k^*)$ is defined by

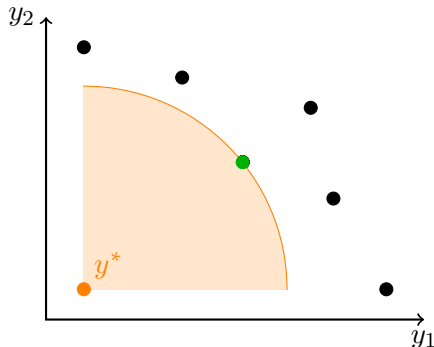
$$y_i^* = \min_{y \in \mathcal{Y}} y_i \quad \forall i.$$



Definition (Compromise Solution, Yu 1973)

Given a multicriteria optimization problem $\min_{y \in \mathcal{Y}} y$ with the ideal point $y^* \in \mathbb{Q}^k$, the *compromise solution* w.r.t. the norm $\|\cdot\|$ on \mathbb{R}^k is

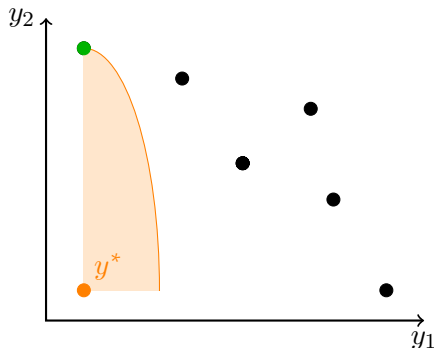
$$y^{\text{CS}} = \min_{y \in \mathcal{Y}} \|y - y^*\|.$$



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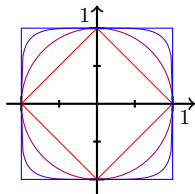


The norms we consider:

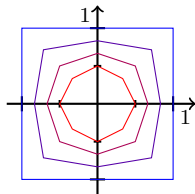
$$\|y\|_p := \left(\sum_{i=1}^k y_i^p \right)^{1/p}, \quad p \in [1, \infty) \quad (\ell^p\text{-Norm})$$

$$\|y\|_\infty := \max_{i=1, \dots, k} y_i \quad (\text{Maximum } (\ell^\infty\text{-})\text{Norm})$$

$$\langle\langle y \rangle\rangle_p := \|y\|_\infty + \frac{1}{p} \|y\|_1, \quad p \in [1, \infty] \quad (\text{Cornered } p\text{-Norm})$$



ℓ^p -Norm



$p = 1$
 $p = 2$
 $p = 5$
 $p = \infty$

Cornered p -Norm

Degree of balancing controlled by adjusting p .

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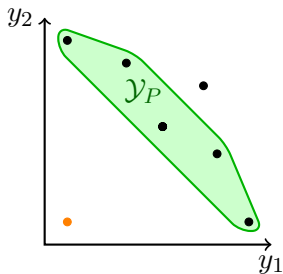
Weighted version: For any norm and $\lambda \in \mathbb{Q}^k, \lambda \geq 0, \lambda \neq 0$:

$$\|y\|^\lambda = \|(\lambda_1 y_1, \lambda_2 y_2, \dots, \lambda_k y_k)\|.$$

Known Properties

Gearhardt 1979:

- ▶ for $p < \infty$ all compromise solutions are Pareto optimal
- ▶ all Pareto optimal solution are a compromise solution, for p big enough



1 Introduction

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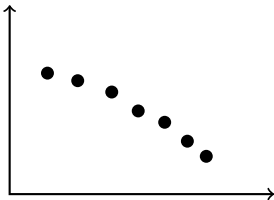
Approximate Pareto sets

Definition (α -approximate Pareto set)

Let \mathcal{Y}_P be the Pareto set of $\min_{y \in \mathcal{Y}} y$, and let $\alpha \geq 1$.

$\mathcal{Y}_\alpha \subseteq \mathcal{Y}$ is an α -approximate Pareto set if for all $y \in \mathcal{Y}_P$ there is $y' \in \mathcal{Y}_\alpha$ such that

$$y'_i \leq \alpha y_i \quad \forall i = 1, \dots, k$$



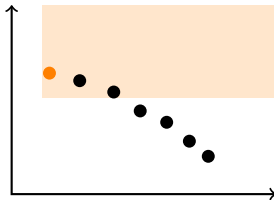
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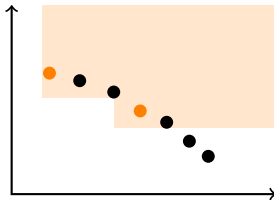
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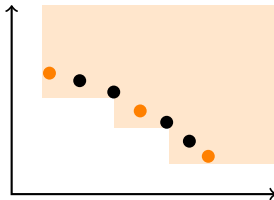
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How to find approximate Pareto sets

Theorem (Papadimitriou&Yannakakis,2000)

$\text{GAP}(y, \alpha)$ tractable for all $y \in \mathbb{Q}^k$
 $\Rightarrow \alpha^2$ -approximation for the Pareto set.

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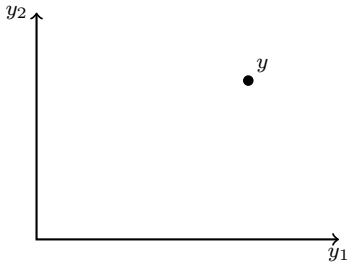
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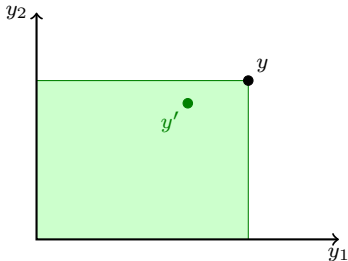
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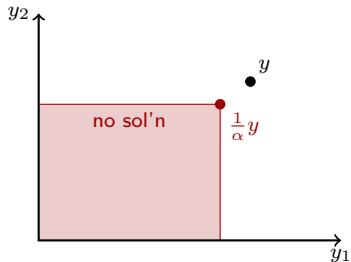
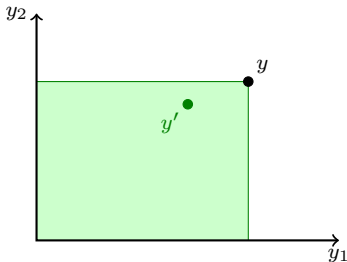
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Approximate Pareto sets \Leftrightarrow approximate CS

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Relate objective value to size of the vectors:

$$\min_{y \in \mathcal{Y}} \underbrace{\|y - y^*\| + \|y^*\|}_{r(y)}$$

We call $r(y)$ the *relative distance*.

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α -approximation of the Pareto set

\Rightarrow α -approximation for $\min_{y \in \mathcal{Y}} r(y)$.

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\Rightarrow $\text{GAP}(y, \beta)$ tractable for all $y \in \mathbb{Q}^k$, $\beta \in \Theta(\alpha)$.

\Rightarrow β^2 -approximation for the Pareto set.

Summary

approximate Pareto set
 ↓ ↑
approximate compromise solutions

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Approximation Algorithms:

- ▶ Approximations based on LP-relaxations
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Thank you for your attention.

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