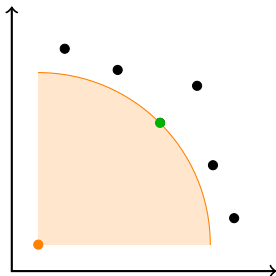


Reference Point Methods and Approximation in Multicriteria Optimization

C. Büsing, **Kai-Simon Goetzmann**, J. Matuschke and S. Stiller



OR Hannover, September 05, 2012

Multicriteria Optimization

The screenshot shows a route planning application interface. On the left, there are controls for route calculation and a list of suggested routes. The main part of the image is a map of Central Germany with a highlighted route from Berlin to Hannover.

Route Calculation Interface:

- Buttons: **Route berechnen**, **Meine Orte**, **Print**, **Fullscreen**
- Mode: **Car** (selected), **Pedestrian**
- Start (A): **Straße des 17. Juni 136, Berlin**
- Destination (B): **Welfengarten 1, 30167 Hannover**
- Buttons: **Ziel hinzufügen - Optionen anzeigen**, **ROUTE BERECHNEN**

Vorgeschlagene Routen (Suggested Routes):

Route	Distance	Duration	Notes
A2	284 km	2 Stunden 48 Minuten	Bei aktueller Verkehrslage: 3 Stunden 2 Minuten
B79 und A2	338 km	3 Stunden 53 Minuten	Keine Verkehrsinformationen

Route nach Welfengarten 1, 30167 Hannover

The map shows a route starting in Berlin (marked with a green 'A') and ending in Hannover (marked with a green 'B'). The route is highlighted in blue and passes through major cities like Magdeburg and Braunschweig. The map includes various road types, landmarks, and a compass.

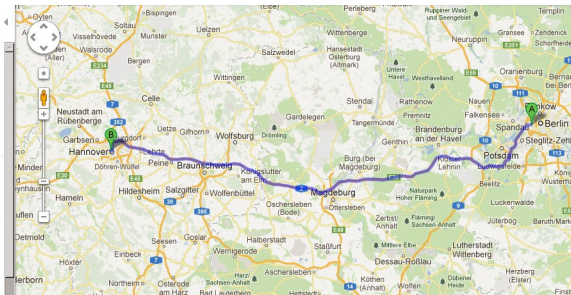
Multicriteria Optimization

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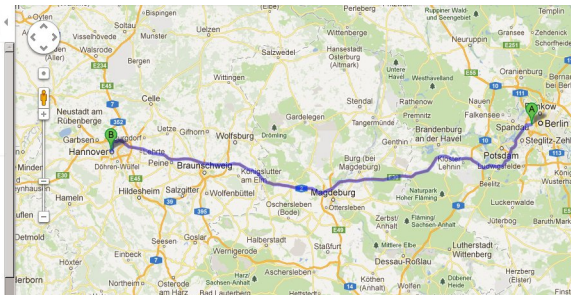
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Multicriteria Optimization

Route berechnen Meine Orte

A Straße des 17. Juni 136, Berlin

B Welfengarten 1, 30167 Hannover

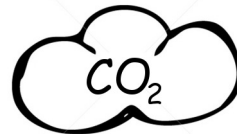
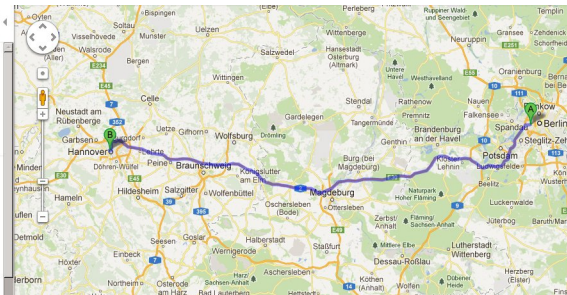
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ROUTE BERECHNEN

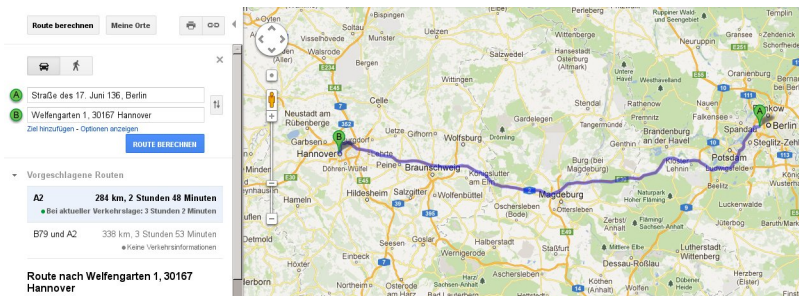
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Multicriteria Optimization



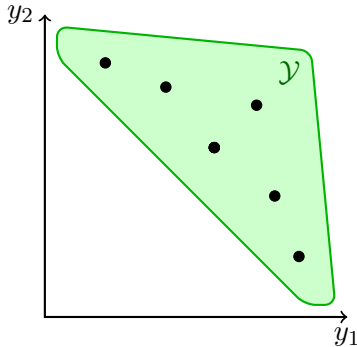
$$\min\{y : y \in \mathcal{Y}\}$$

$$\text{where } \mathcal{Y} \subseteq \mathbb{Z}^k$$

Pareto Optimality

Definition

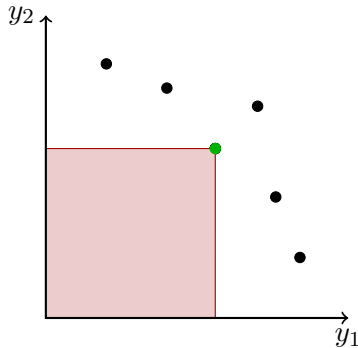
A solution $y \in \mathcal{Y}$ of $\min_{y \in \mathcal{Y}} y$ is *Pareto optimal* if there is no $y' \in \mathcal{Y} \setminus \{y\}$ with $y' \leq y$.



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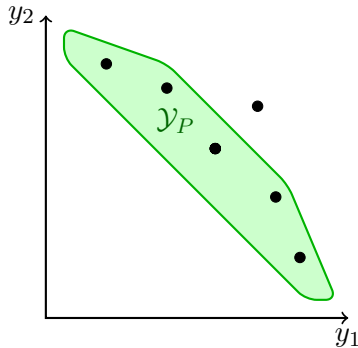
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Reference Point Solutions

Motivation:

- ▶ identify a single, Pareto optimal, balanced solution
- ▶ *reference point methods*:
part of many state-of-the-art MCDM tools,
little theoretical background
- ▶ *A powerful concept*:
all Pareto optimal solutions can be RPS,
approximation of RPS yields approximate Pareto set

1 Introduction

2 Definitions and Notations

3 Approximation

1 Introduction

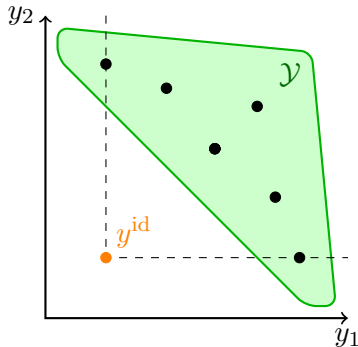
2 Definitions and Notations

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Definition (Ideal Point)

Given a multicriteria optimization problem $\min_{y \in \mathcal{Y}} y$, the *ideal point* $y^{\text{id}} = (y_1^{\text{id}}, \dots, y_k^{\text{id}})$ is defined by

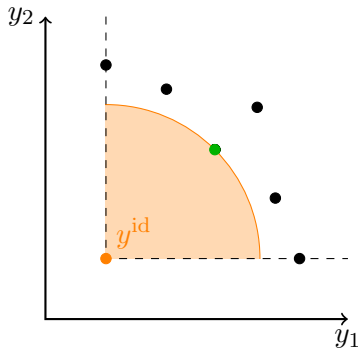
$$y_i^{\text{id}} = \min_{y \in \mathcal{Y}} y_i \quad \forall i.$$



Definition (Compromise Solution, Yu 1973)

Given a multicriteria optimization problem $\min_{y \in \mathcal{Y}} y$ with the ideal point $y^{\text{id}} \in \mathbb{Q}^k$, and a norm $\|\cdot\|$ on \mathbb{R}^k , the *compromise solution* w.r.t. $\|\cdot\|$ is

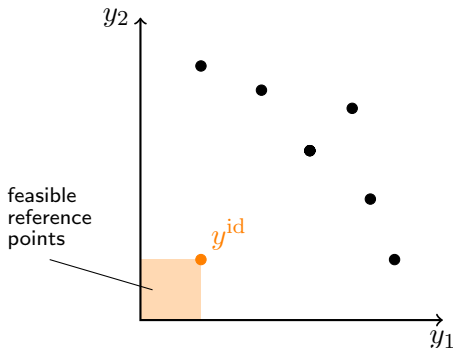
$$y^{\text{cs}} = \min_{y \in \mathcal{Y}} \|y - y^{\text{id}}\|.$$



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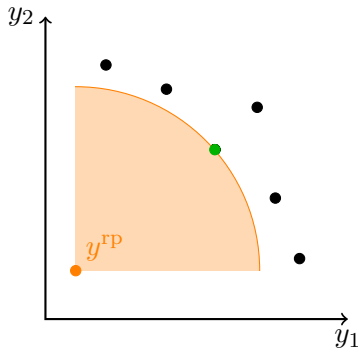
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Definition (Reference Point Solution)

Given a multicriteria optimization problem $\min_{y \in \mathcal{Y}} y$, a **feasible reference point** $y^{\text{rp}} \in \mathbb{Q}^k$, and a norm $\|\cdot\|$ on \mathbb{R}^k , the **reference point solution** w.r.t. $\|\cdot\|$ is

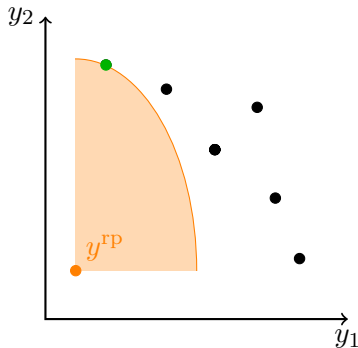
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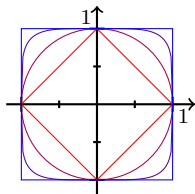


The norms we consider:

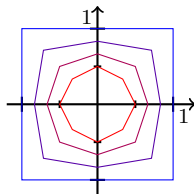
$$\|y\|_p := \left(\sum_{i=1}^k y_i^p \right)^{1/p}, \quad p \in [1, \infty) \quad (\ell^p\text{-Norm})$$

$$\|y\|_\infty := \max_{i=1, \dots, k} y_i \quad (\text{Maximum } (\ell^\infty\text{-Norm)})$$

$$\langle\langle y \rangle\rangle_p := \|y\|_\infty + \frac{1}{p} \|y\|_1, \quad p \in [1, \infty] \quad (\text{Cornered } p\text{-Norm})$$



$\ell^p\text{-Norm}$



Cornered $p\text{-Norm}$

$p = 1$
 $p = 2$
 $p = 5$
 $p = \infty$

Degree of balancing controlled by adjusting p .

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Weighted version: For any norm and $\lambda \in \mathbb{Q}^k, \lambda \geq 0, \lambda \neq 0$:

$$\|y\|^\lambda = \|(\lambda_1 y_1, \lambda_2 y_2, \dots, \lambda_k y_k)\|.$$

General properties: Norms we consider are

- ▶ *monotone* (if $y \geq y'$ then $\|y\| \geq \|y'\|$)
- ▶ *polynomially decidable* ($\|y\| \geq \|y'\|$ can be decided in polynomial time)

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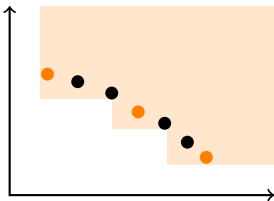
Approximate Pareto sets

Definition (α -approximate Pareto set)

Let \mathcal{Y}_P be the Pareto set of $\min_{y \in \mathcal{Y}} y$, and let $\alpha > 1$.

$\mathcal{Y}_\alpha \subseteq \mathcal{Y}$ is an α -approximate Pareto set if for all $y \in \mathcal{Y}_P$ there is $y' \in \mathcal{Y}_\alpha$ such that

$$y'_i \leq \alpha y_i \quad \forall i = 1, \dots, k$$



How to find approximate Pareto sets

Theorem (Papadimitriou&Yannakakis,2000)

$\text{GAP}(y, \alpha)$ tractable for all $y \in \mathbb{Q}^k$
 $\Rightarrow \alpha^2$ -approximation for the Pareto set.

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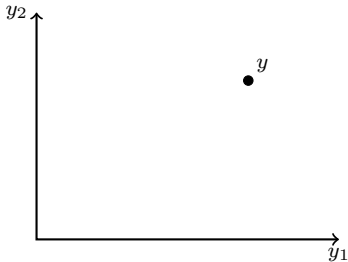
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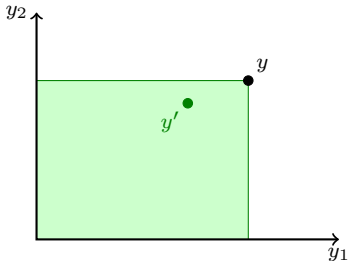
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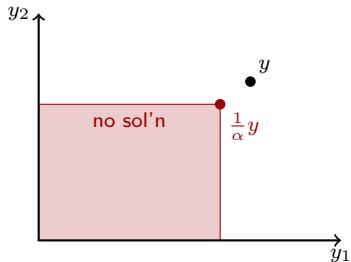
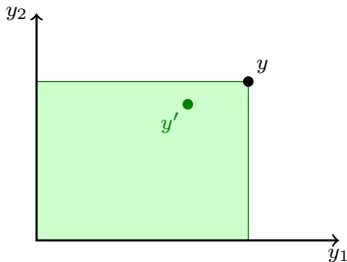
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Approximate Pareto sets \Leftrightarrow approximate RPS

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Relate objective value to size of the vectors:

$$\min_{y \in \mathcal{Y}} \underbrace{\|y - y^{\text{RP}}\| + \|y^{\text{RP}}\|}_{r(y)}$$

We call $r(y)$ the *relative distance*.

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α -approximation of the Pareto set

\Rightarrow α -approximation for $\min_{y \in \mathcal{Y}} r(y)$.

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\Rightarrow $\text{GAP}(y, \beta)$ tractable for all $y \in \mathbb{Q}^k$, $\beta \in \Theta(\alpha)$.

\Rightarrow β^2 -approximation for the Pareto set.

Equivalences of approximability

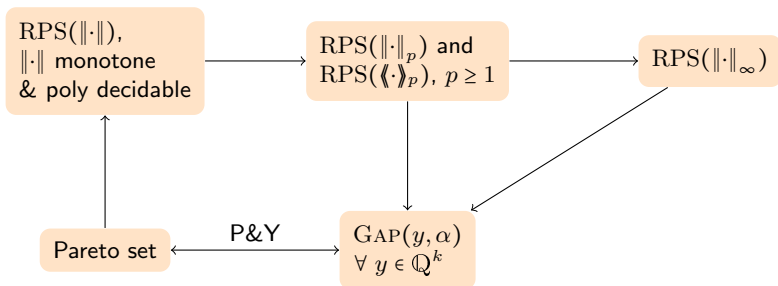
RPS($\|\cdot\|$),
 $\|\cdot\|$ monotone
& poly decidable

RPS($\|\cdot\|_p$) and
RPS($\langle\langle\cdot\rangle\rangle_p$), $p \geq 1$

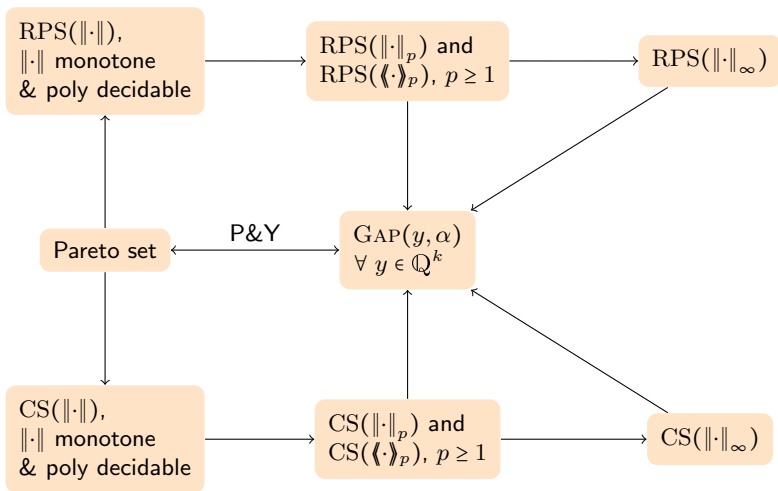
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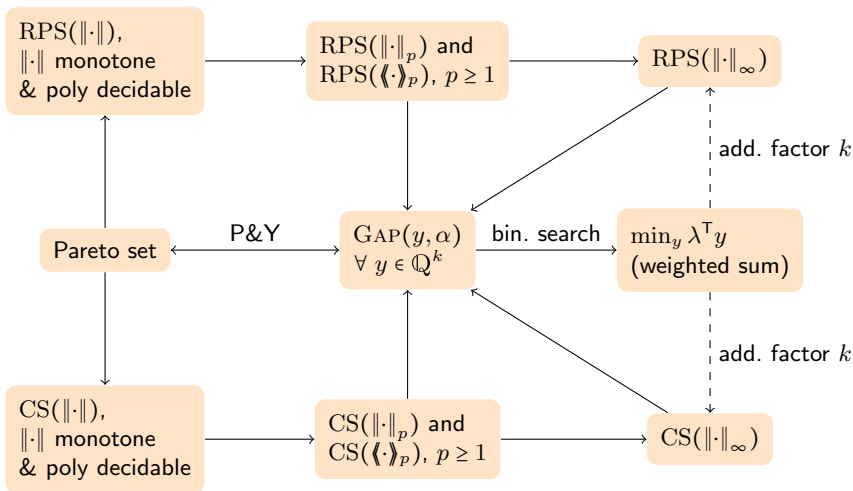
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Approximation through LP-rounding

Extend approximation algorithms based on LP-rounding
(e.g. Set Cover, Scheduling) to compromise programming

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- ▶ rounding procedure $\mathcal{R} : \mathbb{Q}_{\geq 0}^n \rightarrow \mathbb{N}^n$ with

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- ▶ rounding procedure $\mathcal{R} : \mathbb{Q}_{\geq 0}^n \rightarrow \mathbb{N}^n$ with

$$c^\top \mathcal{R}(x) \leq \alpha c^\top x$$

$$\Rightarrow \quad r(C\mathcal{R}(x)) \leq \alpha \cdot r(Cx)$$

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- ▶ scale and round the instance to achieve polynomial running time
- ▶ x^* , \bar{x}^* compromise solutions for original and rounded instance.

$$\|C\bar{x}^* - y^{\text{id}}\|_{\infty} \leq (1 + \varepsilon) \cdot \|Cx^* - y^{\text{id}}\|_{\infty}$$

Approximation through dynamic programming

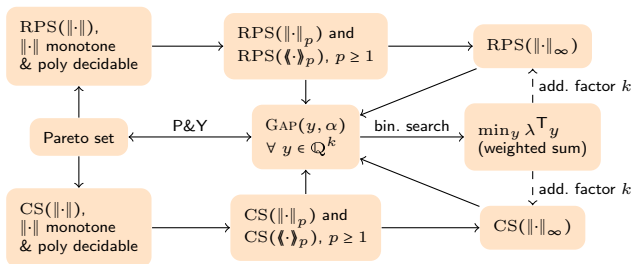
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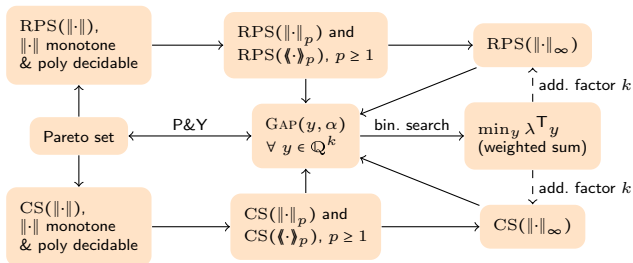
Summary



Approximation Algorithms:

- ▶ Approximations based on LP-relaxations
- ▶ FPTAS based on dynamic programming

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Thank you for your attention.