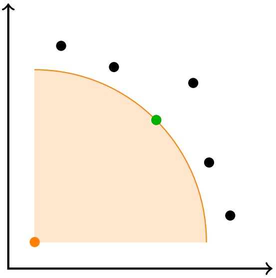
Reference Point Methods

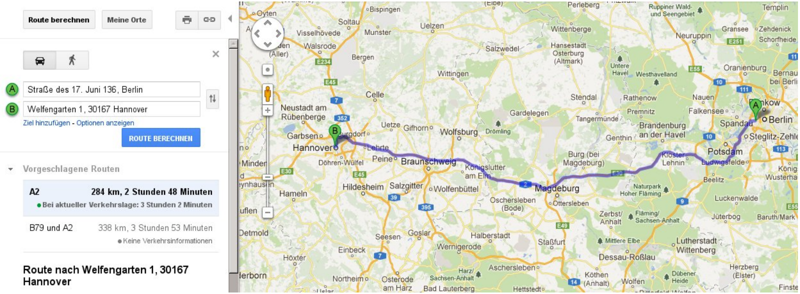
and Approximation in Multicriteria Optimization

C. Büsing, **Kai-Simon Goetzmann**, J. Matuschke and S. Stiller



OR Hannover, September 05, 2012

Multicriteria Optimization



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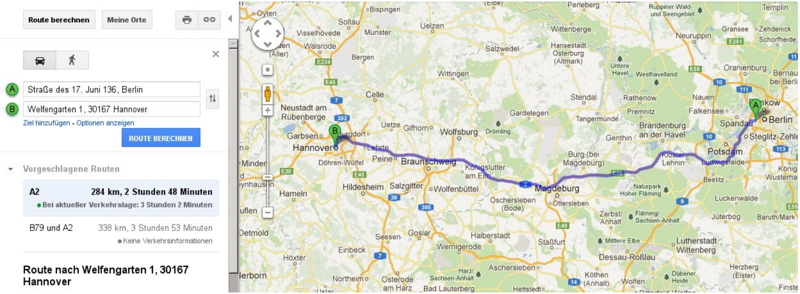

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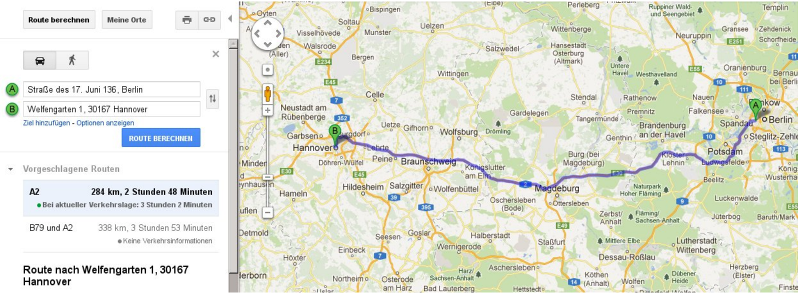

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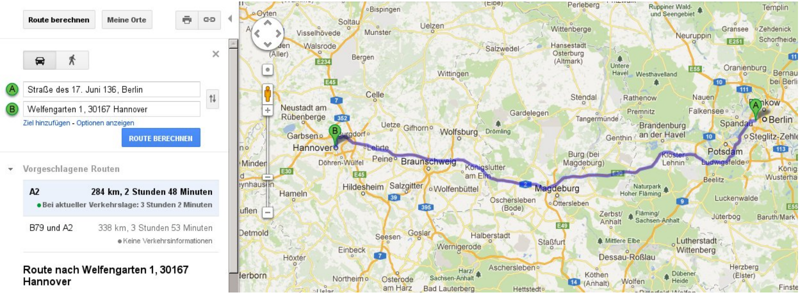

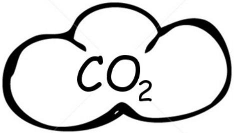

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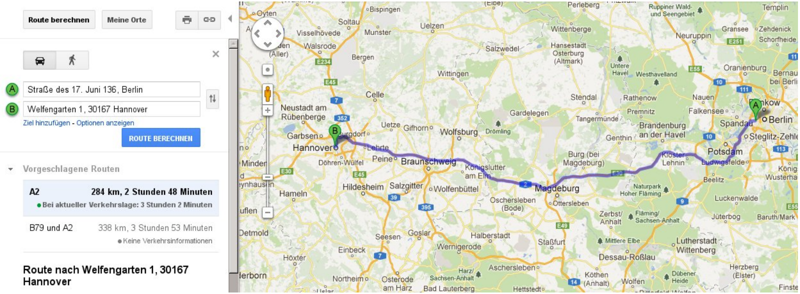
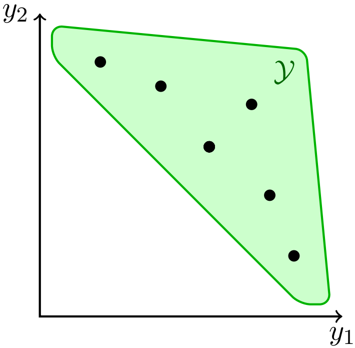


$$\min\{y : y \in \mathcal{Y}\} \qquad \text{where} \quad \mathcal{Y} \subseteq \mathbb{Z}^k$$

Pareto Optimality

Definition

A solution $y \in \mathcal{Y}$ of $\min_{y \in \mathcal{Y}} y$ is *Pareto optimal* if there is no $y' \in \mathcal{Y} \setminus \{y\}$ with $y' \leq y$.



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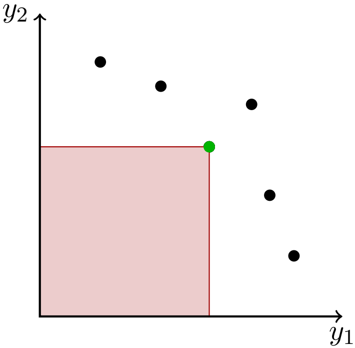

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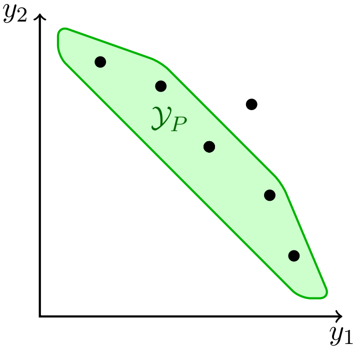


Reference Point Solutions

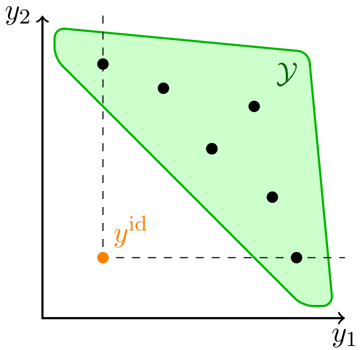
Motivation:

- identify a single, Pareto optimal, balanced solution
- *reference point methods*:
  part of many state-of-the-art MCDM tools,
  little theoretical background
- *A powerful concept*:
  all Pareto optimal solutions can be RPS,
  approximation of RPS yields approximate Pareto set

Definition (Ideal Point)

Given a multicriteria optimization problem $\min_{y \in \mathcal{Y}} y$, the *ideal point* $y^{\text{id}} = (y_1^{\text{id}}, \dots, y_k^{\text{id}})$ is defined by

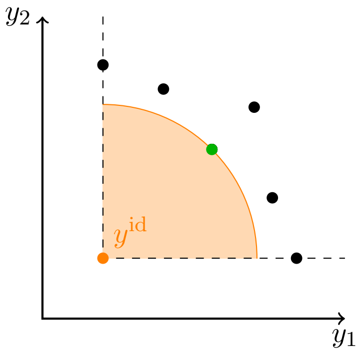
$$y_i^{\text{id}} = \min_{y \in \mathcal{Y}} y_i \qquad \forall\ i.$$



Definition (Compromise Solution, Yu 1973)

Given a multicriteria optimization problem $\min_{y \in \mathcal{Y}} y$ with the ideal point $y^{\mathrm{id}} \in \mathbb{Q}^k$, and a norm $\|\cdot\|$ on $\mathbb{R}^k$, the *compromise solution* w.r.t. $\|\cdot\|$ is

$$y^{\mathrm{cs}} = \min_{y \in \mathcal{Y}} \left\| y - y^{\mathrm{id}} \right\| .$$



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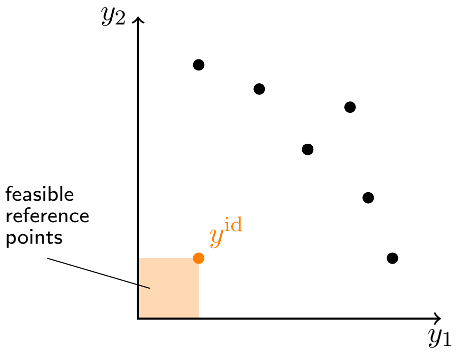

Definition (Reference Point Solution)

Given a multicriteria optimization problem $\min_{y \in \mathcal{Y}} y$, a feasible reference point $y^{\mathrm{rp}} \in \mathbb{Q}^k$, and a norm $\|\cdot\|$ on $\mathbb{R}^k$, the *reference point solution* w.r.t. $\|\cdot\|$ is

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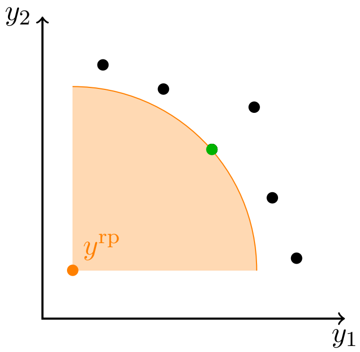

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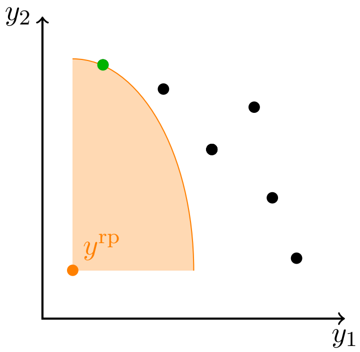
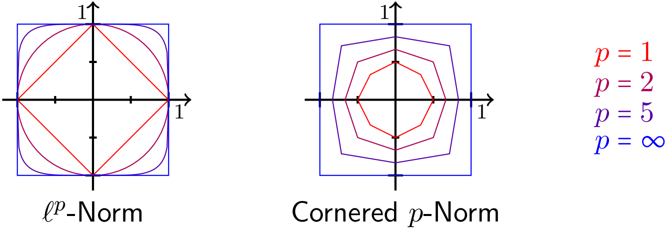


The norms we consider:

$$\|y\|_p := \left(\sum_{i=1}^{k} y_i^p\right)^{1/p}, \quad p \in [1, \infty) \qquad (\ell^p\text{-Norm})$$

$$\|y\|_\infty := \max_{i=1,\dots,k} y_i \qquad (\text{Maximum } (\ell^\infty\text{-})\text{Norm})$$

$$\langle\!\langle y \rangle\!\rangle_p := \|y\|_\infty + \frac{1}{p}\|y\|_1, \quad p \in [1, \infty] \qquad (\textit{Cornered } p\text{-Norm})$$



Degree of balancing controlled by adjusting p.

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Weighted version: For any norm and $\lambda \in \mathbb{Q}^k, \lambda \geq 0, \lambda \neq 0:$

$$\|y\|^\lambda = \|(\lambda_1 y_1, \lambda_2 y_2, \dots, \lambda_k y_k)\|.$$

General properties: Norms we consider are

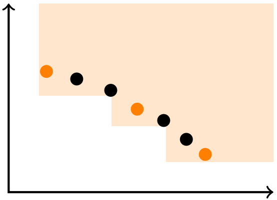
- *monotone* (if $y \geq y'$ then $\|y\| \geq \|y'\|$)
- *polynomially decidable* ($\|y\| \geq \|y'\|$ can be decided in polynomial time)

Approximate Pareto sets

Definition (α-approximate Pareto set)

Let $\mathcal{Y}_P$ be the Pareto set of $\min_{y \in \mathcal{Y}} y$, and let $\alpha > 1$.
$\mathcal{Y}_\alpha \subseteq \mathcal{Y}$ is an *α-approximate Pareto set* if for all $y \in \mathcal{Y}_P$ there is $y' \in \mathcal{Y}_\alpha$ such that

$$y'_i \leq \alpha y_i \quad \forall i = 1, \ldots, k$$



How to find approximate Pareto sets

Theorem (Papadimitriou&Yannakakis,2000)

$\textsc{Gap}(y, \alpha)$ *tractable for all* $y \in \mathbb{Q}^k$
$\Rightarrow \quad \alpha^2$*-approximation for the Pareto set.*

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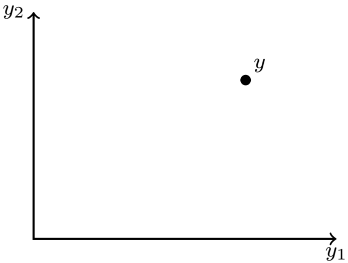

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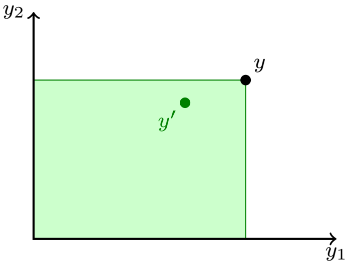

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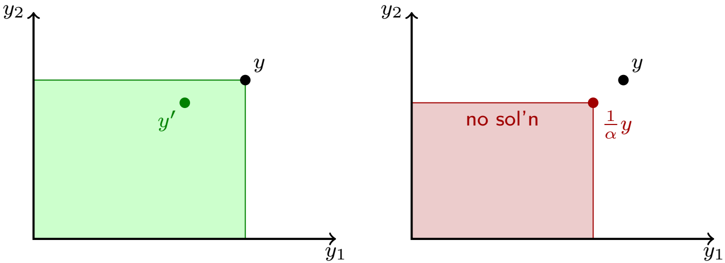

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Relate objective value to size of the vectors:

$$\min_{y \in \mathcal{Y}} \underbrace{\|y - y^{\text{rp}}\| + \|y^{\text{rp}}\|}_{r(y)}$$

We call $r(y)$ the *relative distance*.

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α-approximation of the Pareto set
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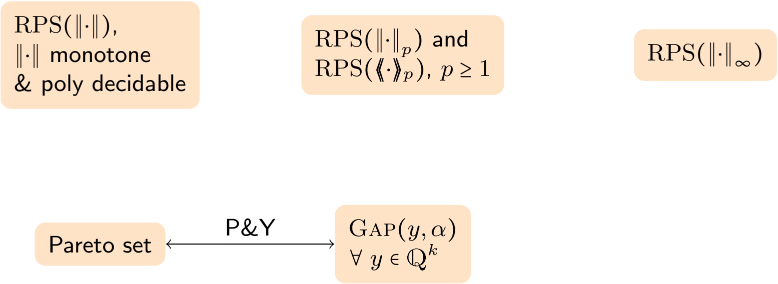
α-approximation for $\min_{y \in \mathcal{Y}} r(y)$
$\Rightarrow$ *$\textsc{Gap}(y, \beta)$ tractable for all $y \in \mathbb{Q}^k$, $\beta \in \Theta(\alpha)$.*
$\Rightarrow$ *β^2-approximation for the Pareto set.*

Equivalences of approximability



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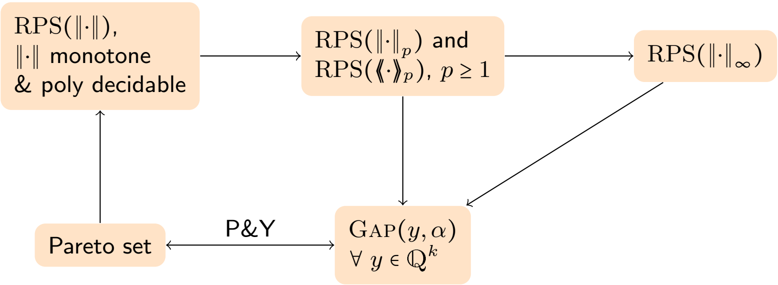

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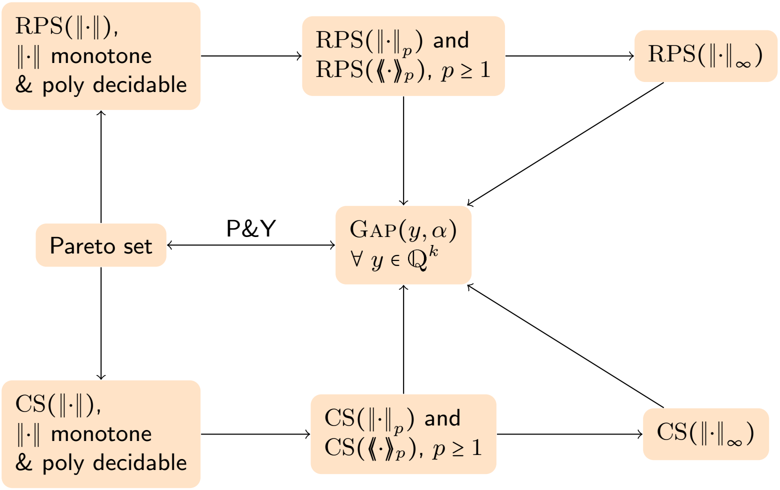

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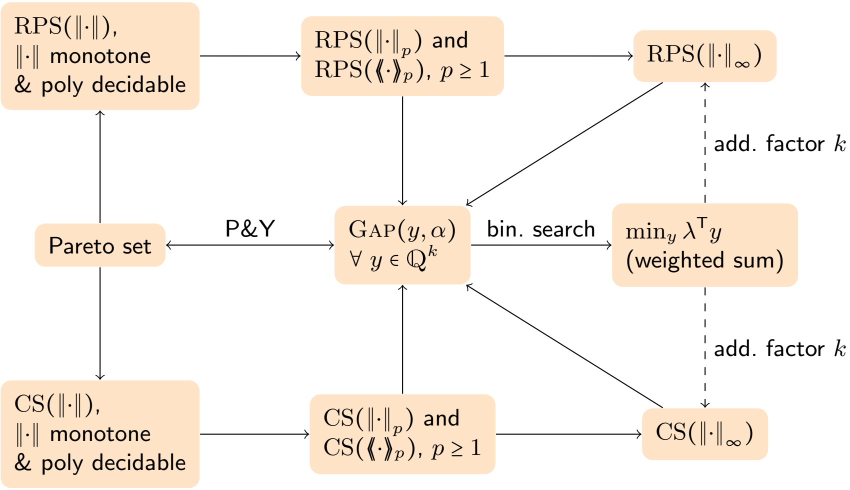

Approximation through LP-rounding

Extend approximation algorithms based on LP-rounding (e.g. Set Cover, Scheduling) to compromise programming

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- rounding procedure $\mathcal{R} : \mathbb{Q}^n_{\geq 0} \to \mathbb{N}^n$ with

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$$c^\top \mathcal{R}(x) \le \alpha c^\top x$$

$$\Rightarrow \quad r(C\mathcal{R}(x)) \le \alpha \cdot r(Cx)$$

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- upper and lower bounds on optimal value
- scale and round the instance to achieve polynomial running time
- $x^*, \overline{x}^*$ compromise solutions for original and rounded instance.

$$\left\| C\overline{x}^* - y^{\mathrm{id}} \right\|_\infty \leq (1+\varepsilon) \cdot \left\| Cx^* - y^{\mathrm{id}} \right\|_\infty$$

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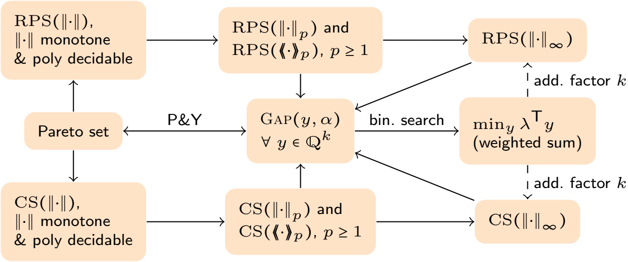
$$\langle\!\langle C\overline{x}^* - y^{\mathrm{rp}} \rangle\!\rangle_p + \langle\!\langle y^{\mathrm{rp}} \rangle\!\rangle_p \leq (1+\varepsilon) \cdot \left(\langle\!\langle Cx^* - y^{\mathrm{rp}} \rangle\!\rangle_p + \langle\!\langle y^{\mathrm{rp}} \rangle\!\rangle_p \right)$$

Summary



Approximation Algorithms:

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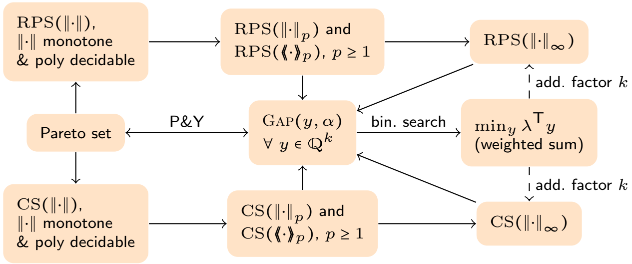
- FPTAS based on dynamic programming

Summary



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Thank you for your attention.