Compromise Solutions in Multicriteria Optimization

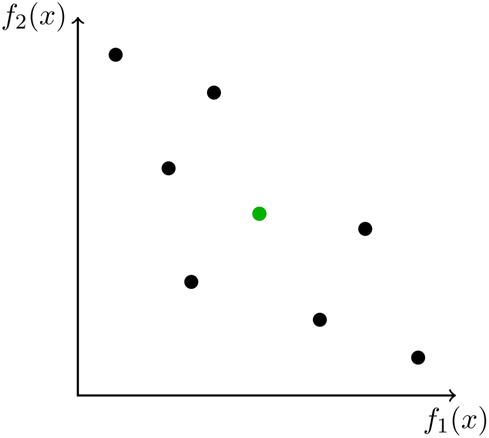
Kai-Simon Goetzmann
(Joint work with Christina Büsing and Jannik Matuschke)

OR 2011 – August 31

Multicriteria Optimization

Pareto optimal solutions:

No objective can be improved without worsening another.



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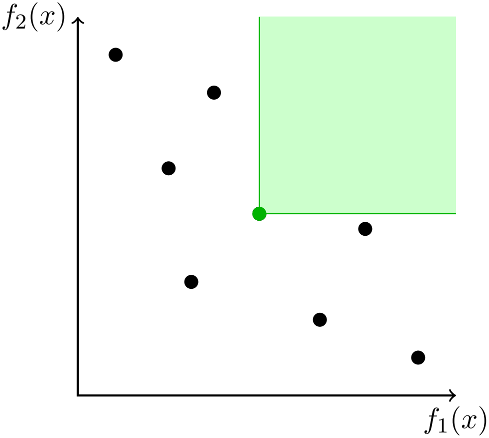

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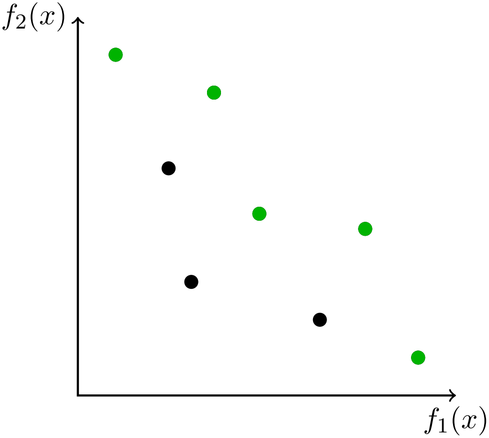
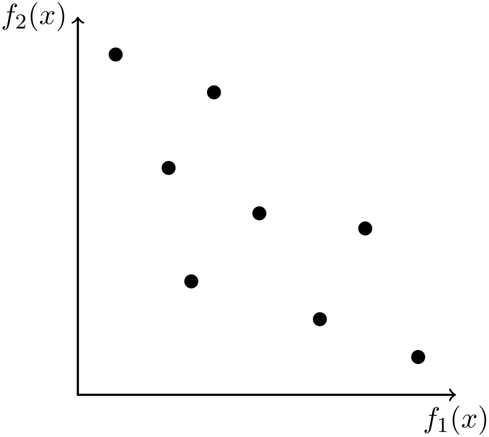


Avoiding too many solutions

Scalarization:

Maximize (weighted) sum of all objectives.



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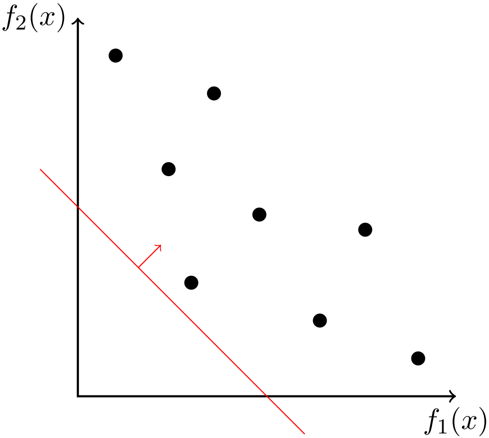

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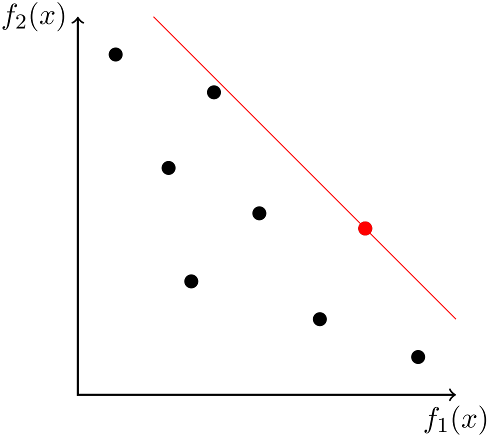

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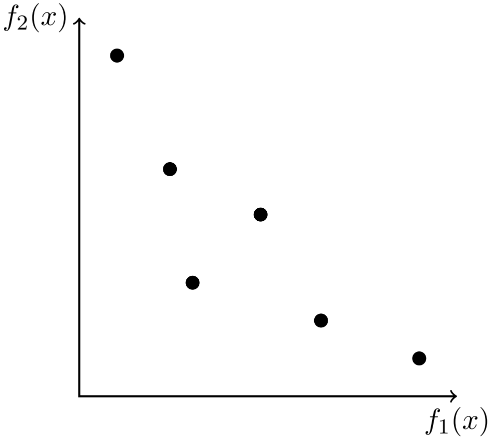

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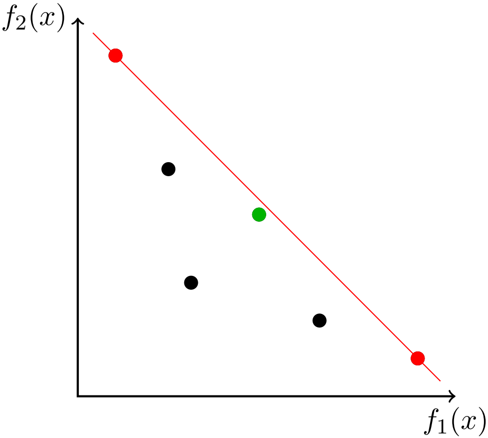
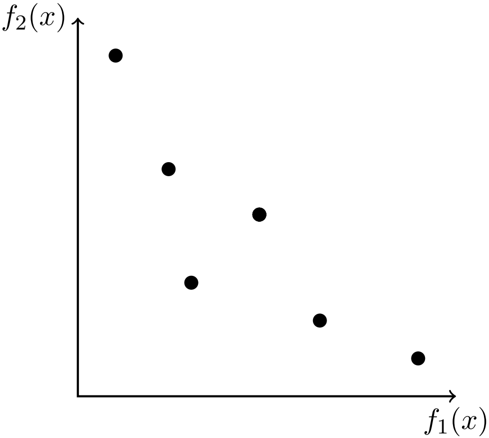


Our Answer

Compromise Solutions:

Minimize distance to ideal point (w.r.t. some metric).



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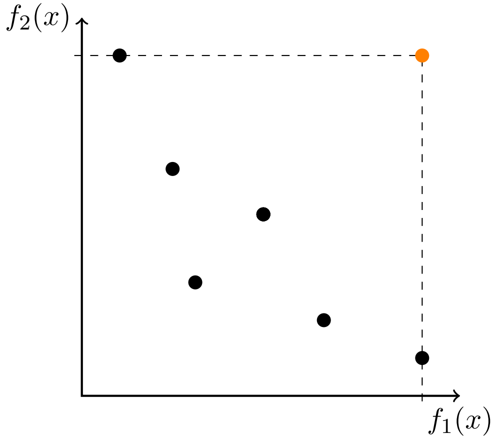

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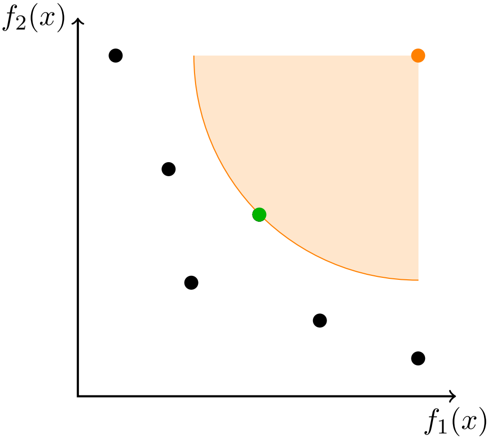
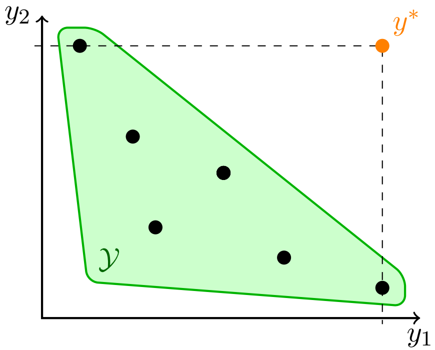


Definition (Ideal Point)

Given a multicriteria optimization problem $\max_{y \in \mathcal{Y}} y$, the *ideal point* $y^* = (y_1^*, \ldots, y_k^*)$ is defined by

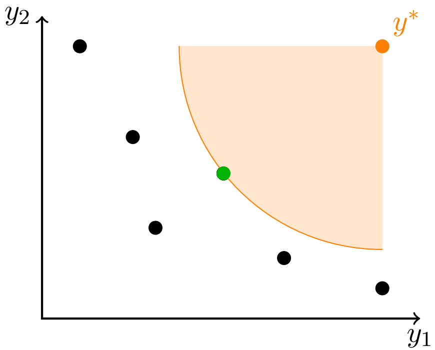
$$y_i^* = \max_{y \in \mathcal{Y}} y_i \qquad \forall\ i.$$



Definition (Compromise Solution, Yu 1973)

Given a multicriteria optimization problem $\max_{y\in\mathcal{Y}} y$ with the ideal point $y^* \in \mathbb{Q}^k$, the *Compromise Solution* w.r.t. the norm $\|\cdot\|$ on $\mathbb{Q}^k$ is

$$y^{\mathsf{CS}} = \min_{y\in\mathcal{Y}} \|y^* - y\| .$$



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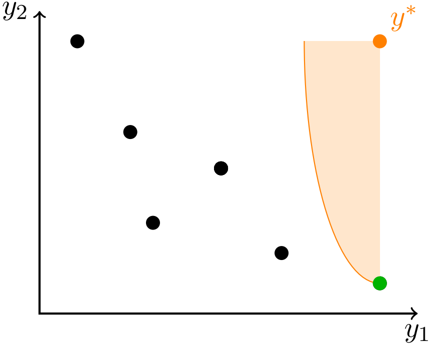
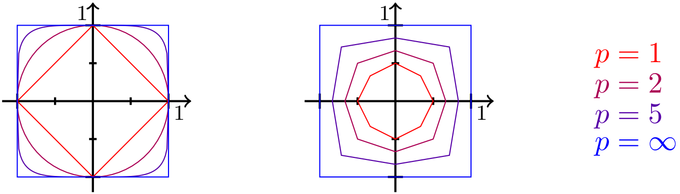


The norms we consider:

$$\|y\|_p := \left(\sum_{i=1}^{k} y_i^p\right)^{1/p}, \quad p \in [1,\infty) \qquad (\ell^p\text{-Norm})$$

$$\|y\|_\infty := \max_{i=1,\dots,k} y_i \qquad (\text{Maximum } (\ell^\infty\text{-})\text{Norm})$$

$$|||y|||_p := \|y\|_\infty + \frac{1}{p}\|y\|_1, \quad p \in [1,\infty] \qquad (\textit{Cornered } p\text{-Norm})$$



ℓ^p-Norm | Cornered p-Norm

Degree of balancing controlled by adjusting p.

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Weighted version: For any norm and $\lambda \in \mathbb{Q}^k, \lambda \geq 0, \lambda \neq 0$:

$$\|y\|^\lambda = \|(\lambda_1 y_1, \lambda_2 y_2, \dots, \lambda_k y_k)\|.$$

Known Properties

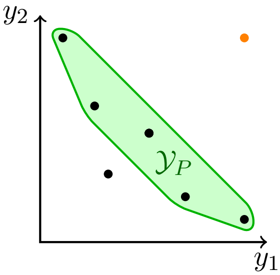
Notation: For a family of norms $\|\cdot\|_p, p \in [1,\infty]$, define

$$\text{CS}(\lambda, p) := \{\text{Compromise Solution w.r.t. } \|\cdot\|_p^\lambda\}$$

$$\text{CS}(p) := \{\text{CS}(\lambda, p) : \lambda \in \mathbb{Q}^k, \lambda \geq 0, \lambda \neq 0\}$$

Gearhardt 1979:

- Pareto optimality: $\text{CS}(p) \subseteq \mathcal{Y}_P$ for $p < \infty$.
- $\mathcal{Y}$ discrete $\Rightarrow \mathcal{Y}_P = \text{CS}(p)$ for p big enough.



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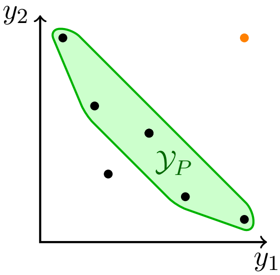


How to choose p

Consider the Cornered Norm $|||\cdot|||_p$.

Assumptions:

- Integrality: $\mathcal{Y} \subset \mathbb{N}_0^k$
- Boundedness:
  $\exists$ polynomial π : $y_i \leq 2^{\pi(|I|)}\ \forall y \in \mathcal{Y}, i = 1, \ldots, k$

Lemma (Büsing,G.,Matuschke)

Let $M := 2^{\pi(|I|)}$. For $p > kM$, it holds that $\mathcal{Y}_P = \text{CS}(p)$.

Remarks:

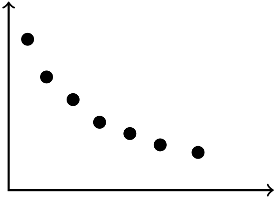
- p can be chosen such that it has polynomial encoding length.
- ℓ^p-norm: $p \gtrsim M \log k$ suffices.

Another way to cope with exponential size of $\mathcal{Y}_P$:

Definition (ε-approximate Pareto set)

Let $\mathcal{Y}_P$ be the Pareto set of a given instance, and let $\varepsilon > 0$. $\mathcal{Y}_{\varepsilon P} \subseteq \mathcal{Y}$ is an *ε-approximate Pareto set* if for all $y \in \mathcal{Y}_P$ there is $y' \in \mathcal{Y}_{\varepsilon P}$ such that

$$y_i \leq (1+\varepsilon) y'_i \quad \forall i = 1, \ldots, k$$



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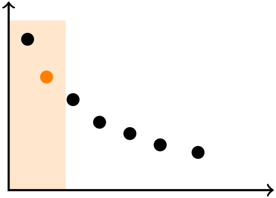

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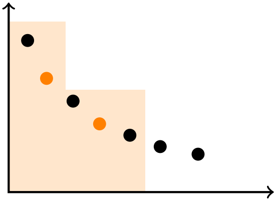

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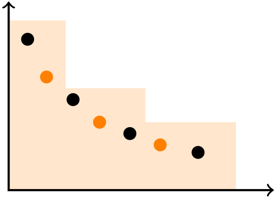

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Theorem (Papadimitriou&Yannakakis,2000)

There always exists an ε-approximate Pareto set with size polynomial in $|I|$ and $1/\varepsilon$.

How to find an ε-approximate Pareto set?

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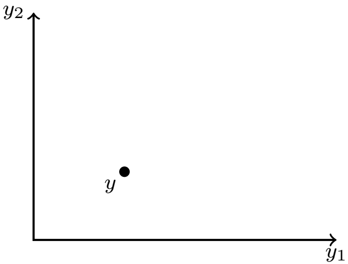
Theorem (Papadimitriou&Yannakakis,2000)

There is an efficient algorithm for constructing an ε-approximate Pareto set if and only if the Gap *problem is tractable.*

Gap problem: Given $y \in \mathbb{Q}^k$ and $\varepsilon > 0$.



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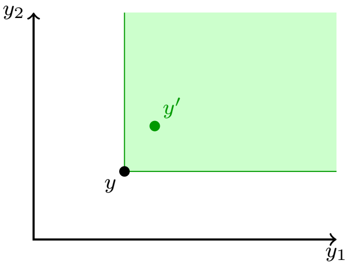

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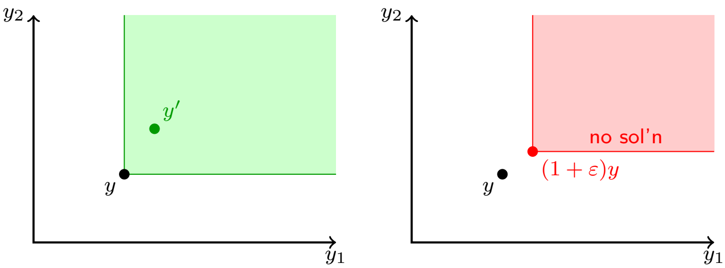

Connection to Compromise Solutions:

Theorem (Büsing,G.,Matuschke)

For $p > \dfrac{kM^2}{\varepsilon}$ *, an FPTAS for* CS(p) *solves the* $\textsc{Gap}$ *problem.*

Remarks:

- p still has polynomial encoding length.
- ℓ^p-norm: $p \gtrsim \frac{M^2 \log k}{\varepsilon}$ suffices.

Proof: Let $y \in \mathbb{Q}^k, \varepsilon > 0$ be given.
First consider $y \in \mathbb{N}^k$, w.l.o.g. $y < y^*$.

For all i, set $\lambda_i := \frac{1}{y_i^* - y_i}$.

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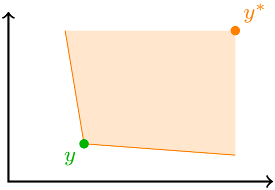

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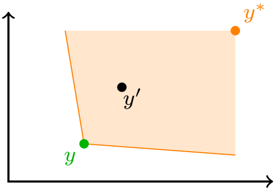


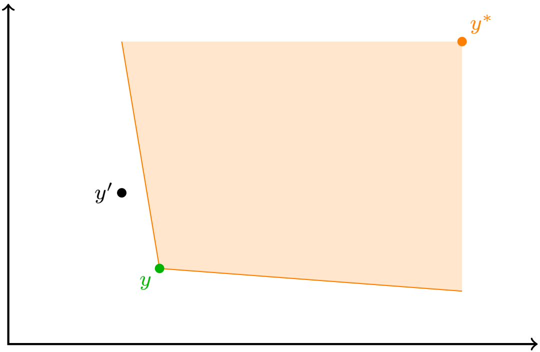
Let y' be returned by the FPTAS for CS(p, λ) for some ε'.

Positive Case:

$$\|y^* - y'\|_p^\lambda \leq \|y^* - y\|_p^\lambda \quad \overset{p>kM}{\Longrightarrow} \quad y' \geq y$$

$$\rightsquigarrow \quad \textsc{Return } y'.$$

Negative Case: $\|y^* - y'\|_p^\lambda > \|y^* - y\|_p^\lambda$



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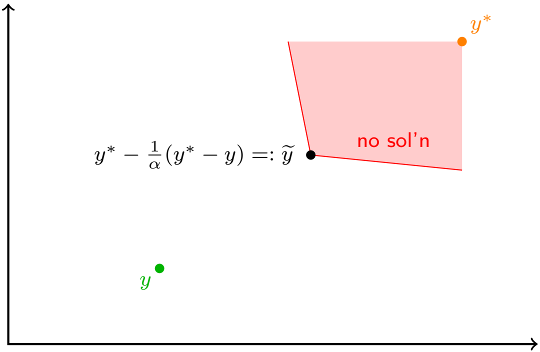

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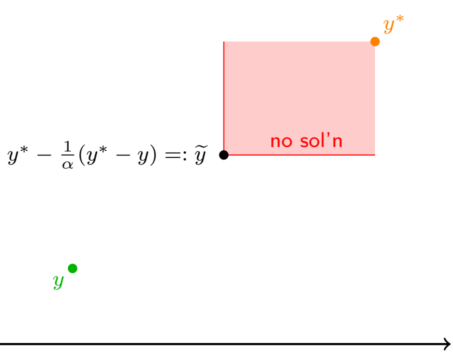

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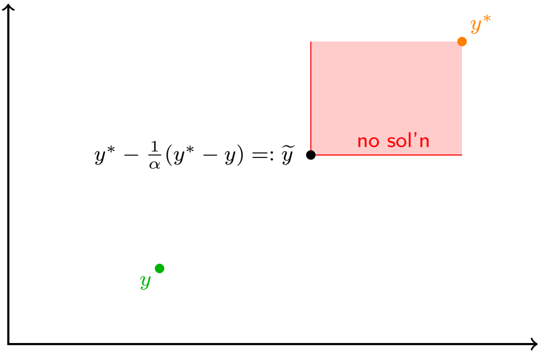

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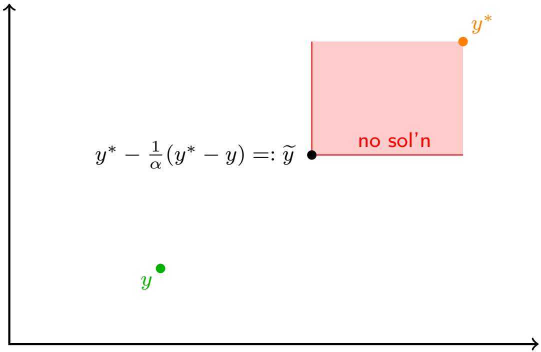

$\Rightarrow$ Need $\tilde{y} \leq (1+\varepsilon)y$.

True for $\alpha = \frac{1}{1-\varepsilon/M}$ $\rightsquigarrow$ $\varepsilon' = 1 - \frac{1}{1-\varepsilon/M}$ ▸ Details

Fractional y: Consider $\lfloor y \rfloor$ instead.

- Similar calculations with more case distinctions,
- slightly smaller value for ε',
- and choosing $p > \frac{kM^2}{\varepsilon}$

enable us to answer $\textsc{Gap}$ also in this case.

Generic Approximation Results

Scalarization: $\max_{y \in \mathcal{Y}} \lambda^\mathsf{T} y$ yields a ρ-approximation to $\mathsf{CS}(p, \lambda)$ with $\rho = \frac{k+kp}{k+p} \leq \min\{k, p+1\}$

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Aissi et al., '06: If there are

- bounds $L \leq$ OPT $\leq U$ with $U \leq \pi_1(|I|)L$
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Corollary

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Problem: Algorithms based on approximating the set of all non-dominated regret-vectors $\approx$ Approximation of Pareto set.

Conclusion

Results:

- Compromise Solutions have nice structural properties; approximating the Pareto set reduces to CS(p).
- FPTAS for some problems, but huge computational effort.
- Simple algorithms (Local Search, Greedy) for MST have no approximation guarantee better than Scalarization (though empirically good).

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Thank you for your attention.


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$$\frac{y_i^* - y_i - \varepsilon y_i}{y_i^* - y_i} \leq 1 - \frac{\varepsilon y_i}{y_i^* - y_i} \leq 1 - \frac{\varepsilon}{M} \qquad (y_i \geq 1)$$

Choosing ε'

W.l.o.g. $y < y^*$ and $\varepsilon < \frac{1}{y_i}$ $\forall i$. For simplicity, assume $y > 0$.

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$$\Rightarrow \quad \alpha = \frac{1}{1 - \varepsilon/M} \quad \text{or} \quad \varepsilon' = 1 - \alpha = 1 - \frac{1}{1 - \varepsilon/M}$$

$\Rightarrow$ For this ε' and $y \in \mathbb{N}^k$, FPTAS for CS(p, λ) answers GAP.

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