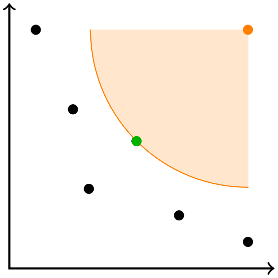
Compromise Solutions

Kai-Simon Goetzmann

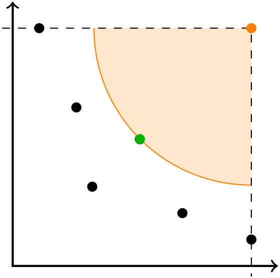
(Joint work with Christina Büsing, Jannik Matuschke and Sebastian Stiller)



MDS status workshop 2012 – February 11

Motivation

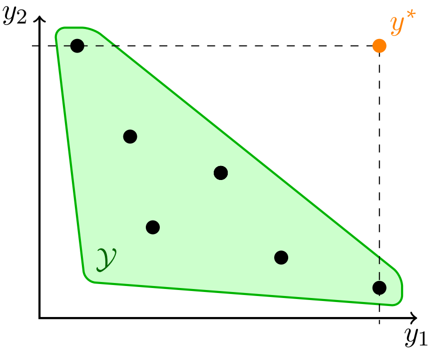
- find *balanced* solutions
- *reference point methods*:
  part of many state-of-the-art MCDM tools,
  little theoretical background



Definition (Ideal Point)

Given a multicriteria optimization problem $\max_{y \in \mathcal{Y}} y$, the *ideal point* $y^* = (y_1^*, \ldots, y_k^*)$ is defined by

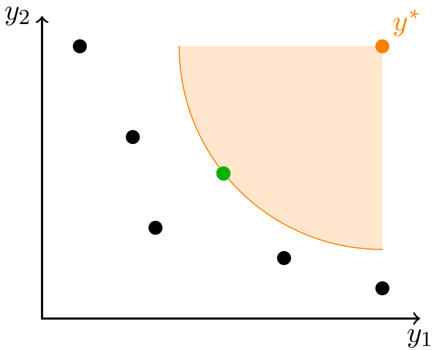
$$y_i^* = \max_{y \in \mathcal{Y}} y_i \qquad \forall\ i.$$



Definition (Compromise Solution, Yu 1973)

Given a multicriteria optimization problem $\max_{y \in \mathcal{Y}} y$ with the ideal point $y^* \in \mathbb{Q}^k$, the *compromise colution* w.r.t. the norm $\|\cdot\|$ on $\mathbb{Q}^k$ is

$$y^{\mathsf{CS}} = \min_{y \in \mathcal{Y}} \|y^* - y\|\,.$$



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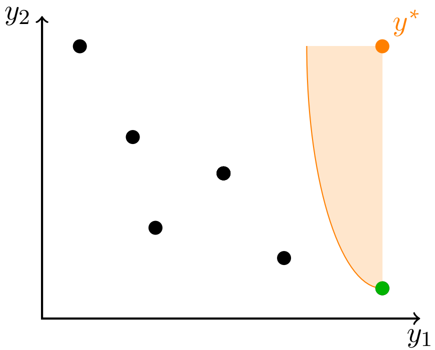
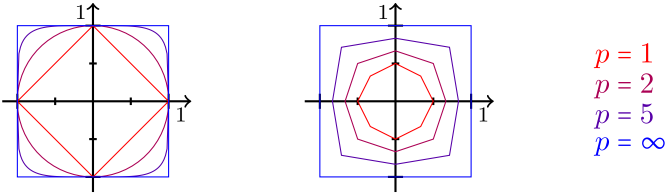


The norms we consider:

$$\|y\|_p := \left(\sum_{i=1}^{k} y_i^p\right)^{1/p}, \quad p \in [1, \infty) \qquad (\ell^p\text{-Norm})$$

$$\|y\|_\infty := \max_{i=1,\dots,k} y_i \qquad (\text{Maximum } (\ell^\infty\text{-})\text{Norm})$$

$$|\!|\!|y|\!|\!|_p := \|y\|_\infty + \frac{1}{p}\|y\|_1, \quad p \in [1, \infty] \qquad (\textit{Cornered } p\text{-Norm})$$



ℓ^p-Norm Cornered p-Norm

Degree of balancing controlled by adjusting p.

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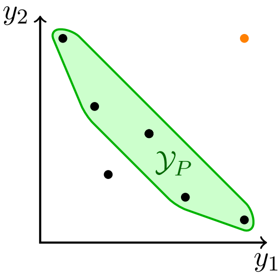
Weighted version: For any norm and $\lambda \in \mathbb{Q}^k, \lambda \geq 0, \lambda \neq 0$:

$$\|y\|^\lambda = \|(\lambda_1 y_1, \lambda_2 y_2, \dots, \lambda_k y_k)\|.$$

Known Properties

Gearhardt 1979:

- for $p < \infty$ all compromise solutions are Pareto optimal
- all Pareto optimal solution are a compromise solution, for p big enough



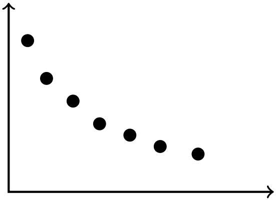
1. Introduction
2. Definitions and Notations
3. **Approximation**

Approximate Pareto sets

Definition (ε-approximate Pareto set)

Let $\mathcal{Y}_P$ be the Pareto set of a given instance, and let $\varepsilon > 0$. $\mathcal{Y}_\varepsilon \subseteq \mathcal{Y}$ is an *ε-approximate Pareto set* if for all $y \in \mathcal{Y}_P$ there is $y' \in \mathcal{Y}_\varepsilon$ such that

$$y_i \leq (1+\varepsilon) y'_i \quad \forall i = 1, \ldots, k$$



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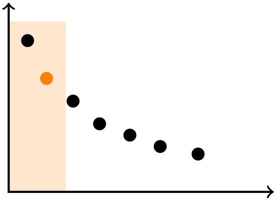

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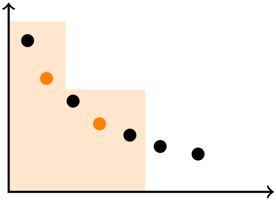

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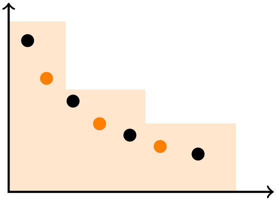

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Theorem (Papadimitriou&Yannakakis,2000)

There always exists an ε-approximate Pareto set with size polynomial in $|I|$ and $1/\varepsilon$.

How to find approximate Pareto sets

Theorem (Papadimitriou&Yannakakis,2000)

There is an efficient algorithm for constructing an ε-approximate Pareto set if and only if the GAP *problem is tractable.*

GAP problem: Given $y \in \mathbb{Q}^k$ and $\varepsilon > 0$.

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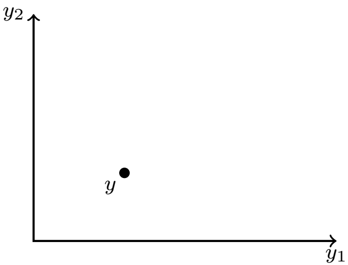

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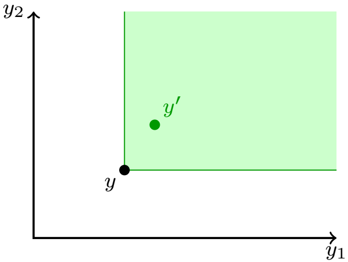

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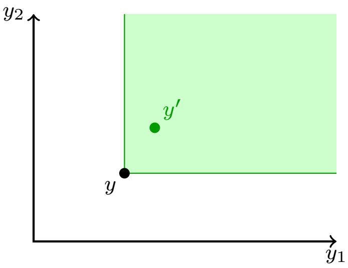

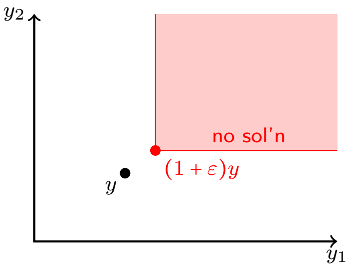

From approximate Pareto sets to approximate CS

Goal: Given an ε-approximate Pareto set,
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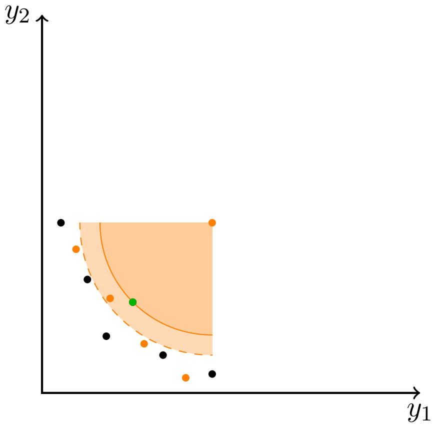

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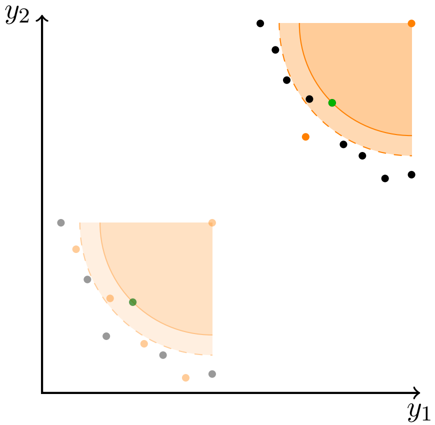

An alternative objective

Replace $\min_{y \in \mathcal{Y}} \|y^* - y\|$ by

$$\max_{y \in \mathcal{Y}} f(y) \; ,$$

where

- level sets are maintained:
  $\|y^* - y\| = \|y^* - y'\| \quad \Rightarrow \quad f(y) = f(y')$

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- $f(0) = 0$
- $f(y^*) = \|y^*\|$

$$\Rightarrow \quad f(y) := \|y^*\| - \|y^* - y\|$$

From approximate Pareto sets to approximate CS

Theorem

If $\mathcal{Y}_\varepsilon$ is an ε-approximate Pareto set for $\max_{y \in \mathcal{Y}} y$, then $\max_{y \in \mathcal{Y}_\varepsilon} f(y)$ yields a $(1+\delta)$-approximation to $\max_{y \in \mathcal{Y}} f(y)$, for some $\delta \in \Theta(\varepsilon)$.

From approximate CS to approximate Pareto set

Goal: Given FPTAS for CS, solve $\textsc{Gap}$ for given $y \in \mathbb{R}^k, \varepsilon > 0$.

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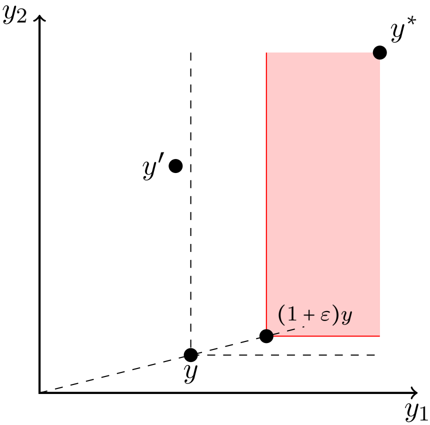

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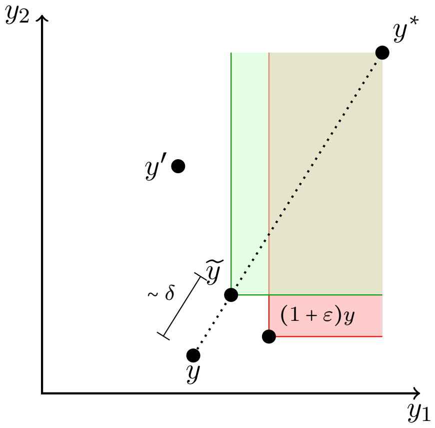

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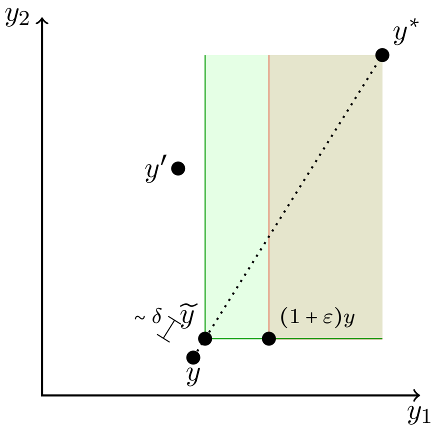
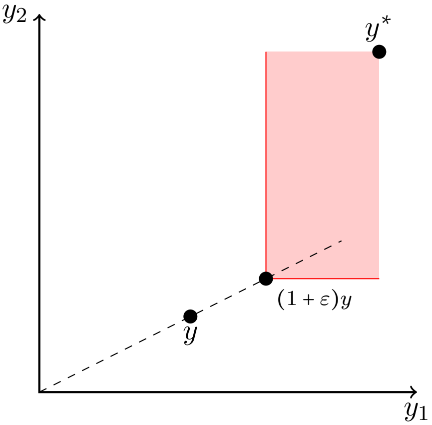


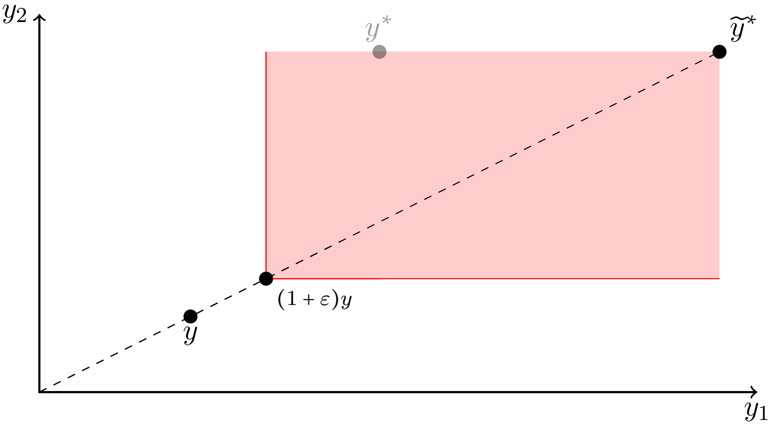
Moving the reference point

Solution: Move the reference point



Moving the reference point

Solution: Move the reference point $\rightsquigarrow$ *super-ideal reference point*



From approximate CS to approximate Pareto set

For $\widetilde{y}^* \geq y^*$, set $\widetilde{f}(y) := \|\widetilde{y}^*\| - \|\widetilde{y}^* - y\|$.

Theorem

If there is an FPTAS for $\max_{y \in \mathcal{Y}} \widetilde{f}(y)$ for any $\widetilde{y}^* \geq y^*$, then the GAP *problem for $\max_{y \in \mathcal{Y}} y$ is tractable.*

Proof

From approximate CS to approximate Pareto set

For $\widetilde{y}^* \geq y^*$, set $\widetilde{f}(y) := \|\widetilde{y}^*\| - \|\widetilde{y}^* - y\|$.

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If $\mathcal{Y}_\varepsilon$ is an ε-approximate Pareto set for $\max_{y \in \mathcal{Y}} y$, then for any $\widetilde{y}^* \geq y^*$, $\max_{y \in \mathcal{Y}_\varepsilon} \widetilde{f}(y)$ yields a $(1+\delta)$-approximation to $\max_{y \in \mathcal{Y}} \widetilde{f}(y)$, for some $\delta \in \Theta(\varepsilon)$.

▸ Proof

Summary & Outlook

approximate Pareto set

ideal point	super-ideal ref pt

approximate compromise solutions

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Future work:

- Approximation algorithms
- Heuristics

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- Application
  Group of *Multicriteria Analysis and Sustainability*,
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Thank you for your attention.

Set $\widetilde{y}^* = C \cdot y$ such that $\widetilde{y}^* \geq y^*$.
Let y' be a $(1+\delta)$-approximation for $\max_{y'' \in \mathcal{Y}} \widetilde{f}(y'')$ with $y' \not\geq y$.

Back

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Let y' be a $(1+\delta)$-approximation for $\max_{y'' \in \mathcal{Y}} \widetilde{f}(y'')$ with $y' \ngeq y$.

$$\begin{aligned} \forall\, y'' \in \mathcal{Y} : \widetilde{f}(y'') &\leq (1+\delta)\widetilde{f}(y') \\ &\leq (1+\delta)\widetilde{f}(y) \\ &= (1+\delta)(\|\widetilde{y}^*\| - \|\widetilde{y}^* - y\|) \\ &= (1+\delta)(C \cdot \|y\| - (C-1)\|y\|) \\ &= (1+\delta)\|y\| \end{aligned}$$

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&= (1+\delta)(C \cdot \|y\| - (C-1)\|y\|) \\
&= (1+\delta)\|y\|
\end{aligned}$$

$$\begin{aligned}
\widetilde{f}((1+\varepsilon)y) &= \left\|\widetilde{y^*}\right\| - \|\widetilde{y}^* - (1+\varepsilon)y\| \\
&= C \cdot \|y\| - (C - (1+\varepsilon))\|y\| \\
&= (1+\varepsilon)\|y\|
\end{aligned}$$

Back

Let $\mathcal{Y}_\varepsilon$ be an ε-approximate Pareto set, $y' = \operatorname{argmax}_{y \in \mathcal{Y}_\varepsilon} \widetilde{f}(y)$.
y^{cs} Pareto optimal
$\Rightarrow \ \exists \bar{y} \in \mathcal{Y}_\varepsilon$ with $\bar{y} \geq (1-\varepsilon)y^{\mathsf{cs}}$, w.l.o.g. $\bar{y} = (1-\varepsilon)y^{\mathsf{cs}}$.
We need

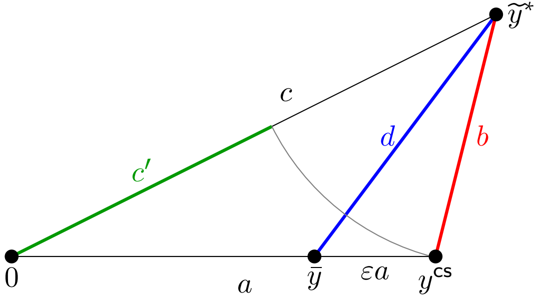
$$\|\widetilde{y}^*\| - \|\widetilde{y}^* - \bar{y}\| = \widetilde{f}(\bar{y}) \geq (1-\delta)\widetilde{f}(y^{\mathsf{cs}}) = (1-\delta)\left(\|\widetilde{y}^*\| - \|\widetilde{y}^* - y^{\mathsf{cs}}\|\right),$$

or equivalently

$$\|\widetilde{y}^* - \bar{y}\| \leq \|\widetilde{y}^* - y^{\mathsf{cs}}\| + \delta\left(\|\widetilde{y}^*\| - \|\widetilde{y}^* - y^{\mathsf{cs}}\|\right).$$

Back

$$\underbrace{\|\widetilde{y}^* - \bar{y}\|}_{d} \leq \underbrace{\|\widetilde{y}^* - y^{\mathsf{cs}}\|}_{b} + \delta \underbrace{\left(\|\widetilde{y}^*\| - \|\widetilde{y}^* - y^{\mathsf{cs}}\|\right)}_{c'}$$



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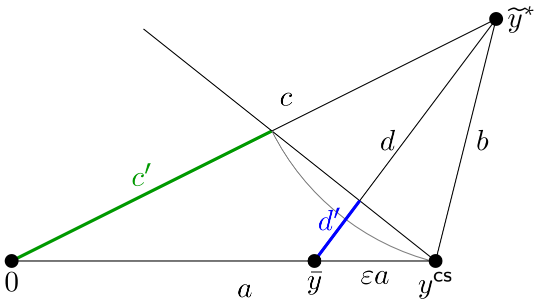

Show $d' \leq \delta c'$.

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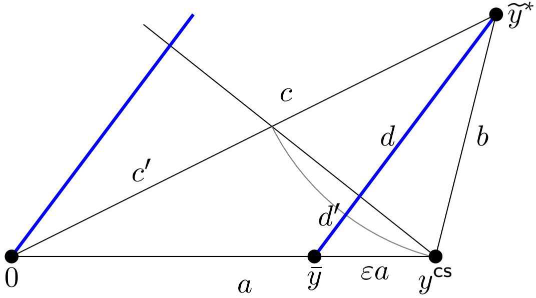

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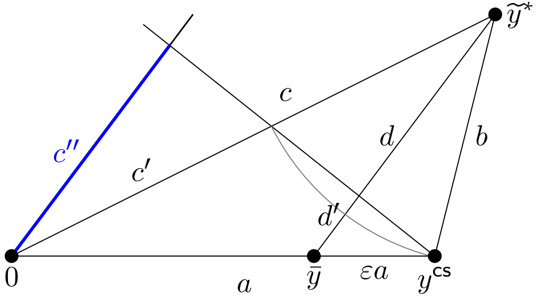

Show $d' \leq \delta c'$.

$$\frac{d'}{\varepsilon a} = \frac{c''}{a}$$

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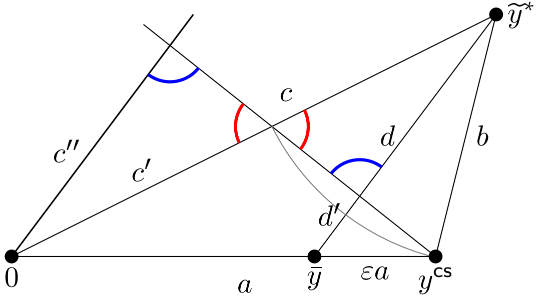

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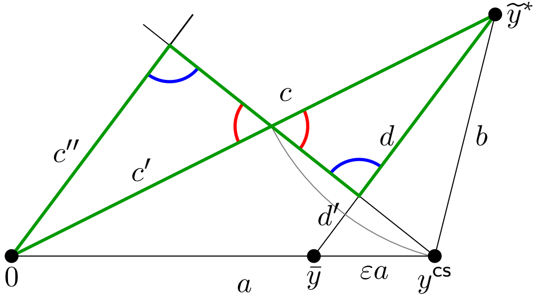


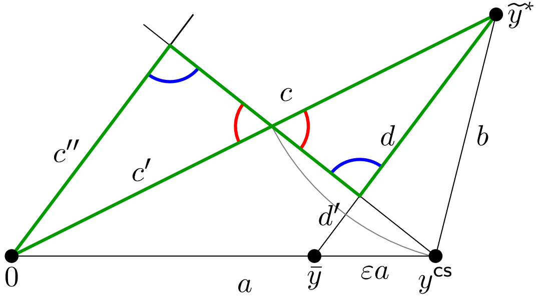
Show $d' \leq \delta c'$.

$$\frac{d'}{\varepsilon a} = \frac{c''}{a}$$

$$\frac{c''}{c'} = \frac{d - d'}{c - c'} = \frac{d - d'}{b} \leq 1$$

▸ Back

$$\underbrace{\|\widetilde{y}^* - \bar{y}\|}_{d} \leq \underbrace{\|\widetilde{y}^* - y^{\text{CS}}\|}_{b} + \delta \underbrace{\left(\|\widetilde{y}^*\| - \|\widetilde{y}^* - y^{\text{CS}}\|\right)}_{c'}$$



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