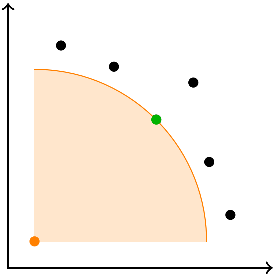
Compromise Solutions

and Approximation in Multicriteria Optimization

C. Büsing, **Kai-Simon Goetzmann**, J. Matuschke and S. Stiller

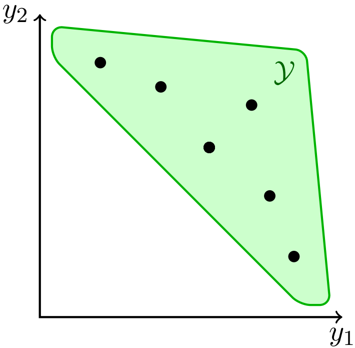


ISMP, August 20, 2012

Multicriteria Optimization

- discrete minimization:

$$\min_{y \in \mathcal{Y}} y, \quad \text{where } \mathcal{Y} \subseteq \mathbb{Z}^k_{\geq 0}$$



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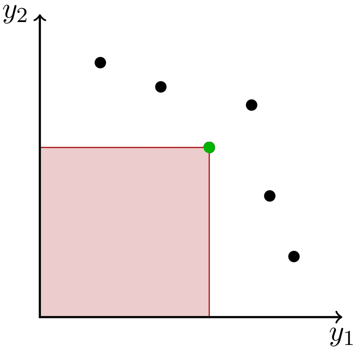

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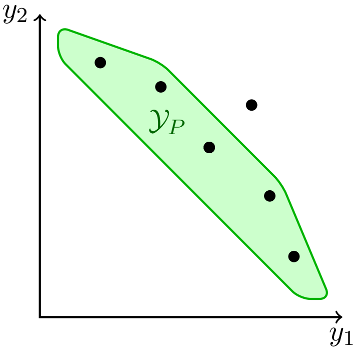
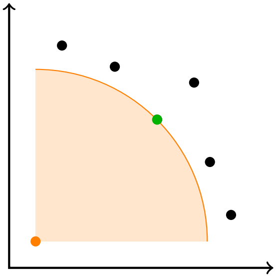


Compromise Programming

Motivation:

- identify a single, Pareto optimal, balanced solution
- *reference point methods*:
  part of many state-of-the-art MCDM tools,
  little theoretical background

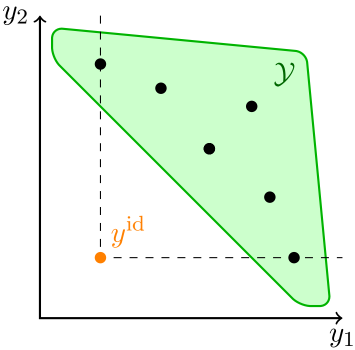


1. Introduction
2. Definitions and Notations
3. Approximation

Definition (Ideal Point)

Given a multicriteria optimization problem $\min_{y \in \mathcal{Y}} y$, the *ideal point* $y^{\text{id}} = (y_1^{\text{id}}, \dots, y_k^{\text{id}})$ is defined by

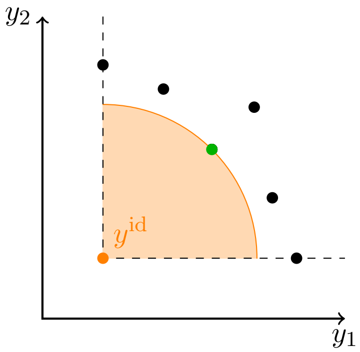
$$y_i^{\text{id}} = \min_{y \in \mathcal{Y}} y_i \qquad \forall\ i.$$



Definition (Compromise Solution, Yu 1973)

Given a multicriteria optimization problem $\min_{y \in \mathcal{Y}} y$ with the ideal point $y^{\mathrm{id}} \in \mathbb{Q}^k$, and a norm $\|\cdot\|$ on $\mathbb{R}^k$, the *compromise solution* w.r.t. $\|\cdot\|$ is

$$y^{\mathrm{cs}} = \min_{y \in \mathcal{Y}} \left\| y - y^{\mathrm{id}} \right\| .$$



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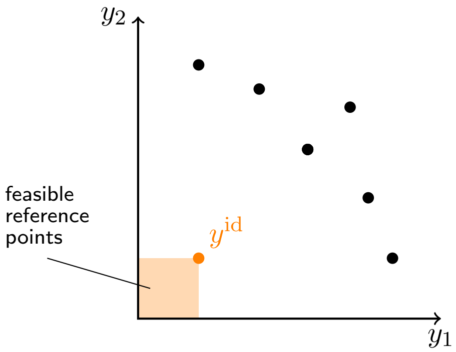
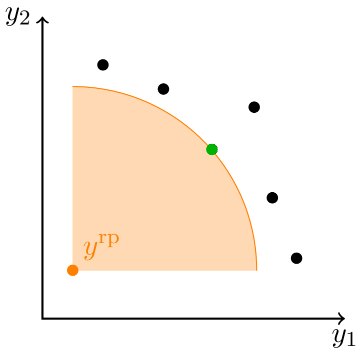


Definition (Reference Point Solution)

Given a multicriteria optimization problem $\min_{y \in \mathcal{Y}} y$, a feasible reference point $y^{\mathrm{rp}} \in \mathbb{Q}^k$, and a norm $\|\cdot\|$ on $\mathbb{R}^k$, the *reference point solution* w.r.t. $\|\cdot\|$ is

$$y^{\mathrm{rps}} = \min_{y \in \mathcal{Y}} \|y - y^{\mathrm{rp}}\| .$$



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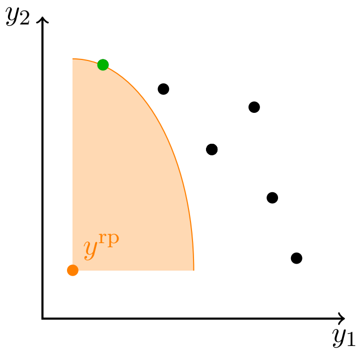
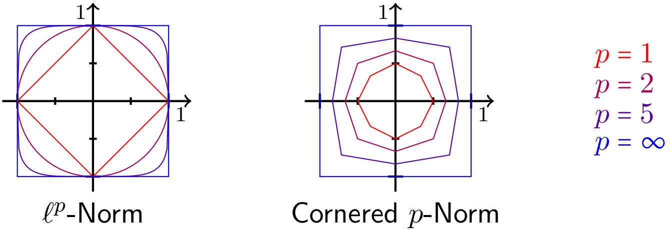


The norms we consider:

$$\|y\|_p := \left(\sum_{i=1}^{k} y_i^p\right)^{1/p}, \quad p \in [1, \infty) \qquad (\ell^p\text{-Norm})$$

$$\|y\|_\infty := \max_{i=1,\dots,k} y_i \qquad (\text{Maximum } (\ell^\infty\text{-})\text{Norm})$$

$$\langle\!\langle y \rangle\!\rangle_p := \|y\|_\infty + \frac{1}{p}\|y\|_1, \quad p \in [1, \infty] \qquad (\textit{Cornered } p\text{-Norm})$$



Degree of balancing controlled by adjusting p.

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Weighted version: For any norm and $\lambda \in \mathbb{Q}^k, \lambda \geq 0, \lambda \neq 0$:

$$\|y\|^\lambda = \|(\lambda_1 y_1, \lambda_2 y_2, \dots, \lambda_k y_k)\|.$$

General properties: Norms we consider are

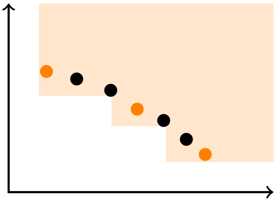
- *monotone* (if $y \geq y'$ then $\|y\| \geq \|y'\|$)
- *polynomially decidable* ($\|y\| \geq \|y'\|$ can be decided in polynomial time)

Approximate Pareto sets

Definition (α-approximate Pareto set)

Let $\mathcal{Y}_P$ be the Pareto set of $\min_{y \in \mathcal{Y}} y$, and let $\alpha > 1$.
$\mathcal{Y}_\alpha \subseteq \mathcal{Y}$ is an α*-approximate Pareto set* if for all $y \in \mathcal{Y}_P$ there is $y' \in \mathcal{Y}_\alpha$ such that

$$y'_i \leq \alpha y_i \quad \forall i = 1, \dots, k$$



How to find approximate Pareto sets

Theorem (Papadimitriou&Yannakakis,2000)

$\textsc{Gap}(y,\alpha)$ *tractable for all* $y \in \mathbb{Q}^k$
$\Rightarrow$ α^2*-approximation for the Pareto set.*

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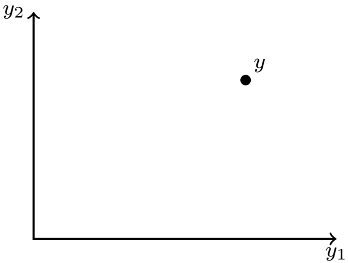

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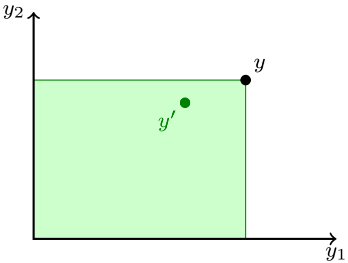

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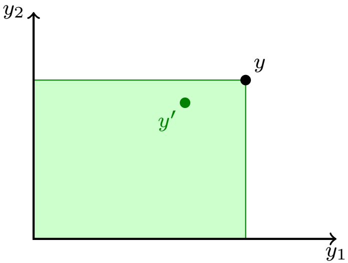

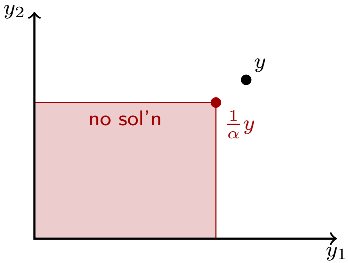

Approximate Pareto sets $\Leftrightarrow$ approximate CS

Approximate Pareto sets ⇔ approximate CS

Relate objective value to size of the vectors:

$$\min_{y \in \mathcal{Y}} \underbrace{\left\| y - y^{\mathrm{id}} \right\| + \left\| y^{\mathrm{id}} \right\|}_{r(y)}$$

We call $r(y)$ the *relative distance*.

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α-approximation of the Pareto set
$\Rightarrow$ *α-approximation for* $\min_{y \in \mathcal{Y}} r(y)$.

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Theorem

α-approximation for $\min_{y \in \mathcal{Y}} r(y)$
$\Rightarrow$ *$\textsc{Gap}(y, \beta)$ tractable for all $y \in \mathbb{Q}^k$, $\beta \in \Theta(\alpha)$.*
$\Rightarrow$ *β^2-approximation for the Pareto set.*

Equivalences of approximability

$\mathrm{CS}(\|\cdot\|)$,
$\|\cdot\|$ monotone
& poly decidable

$\mathrm{CS}(\|\cdot\|_p)$ and
$\mathrm{CS}(\langle\!\langle\cdot\rangle\!\rangle_p)$, $p \geq 1$

$\mathrm{CS}(\|\cdot\|_\infty)$

Pareto set $\longleftrightarrow$ (P&Y) $\textsc{Gap}(y, \alpha)$ $\forall\, y \in \mathbb{Q}^k$

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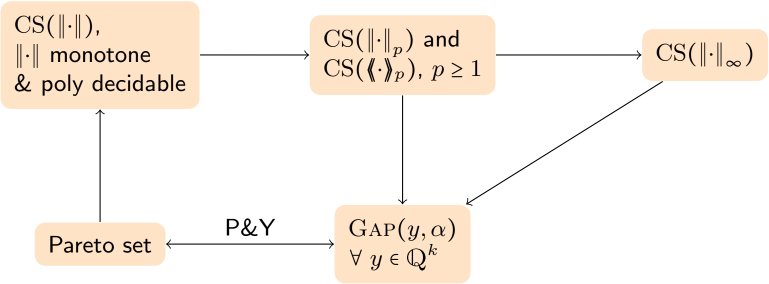

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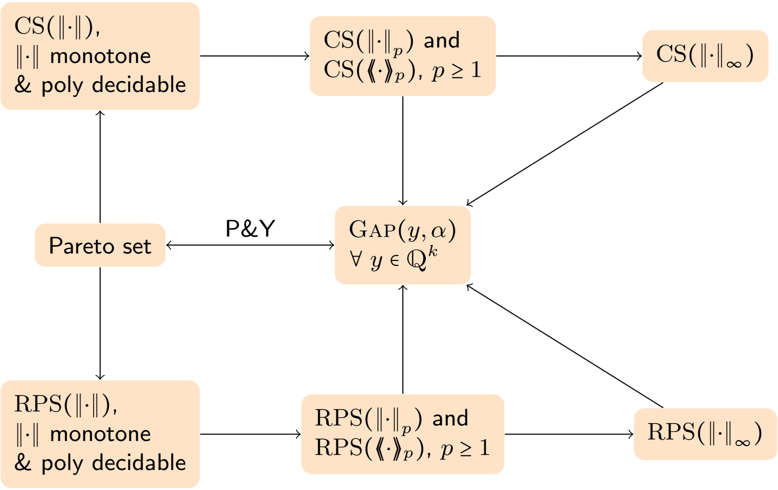

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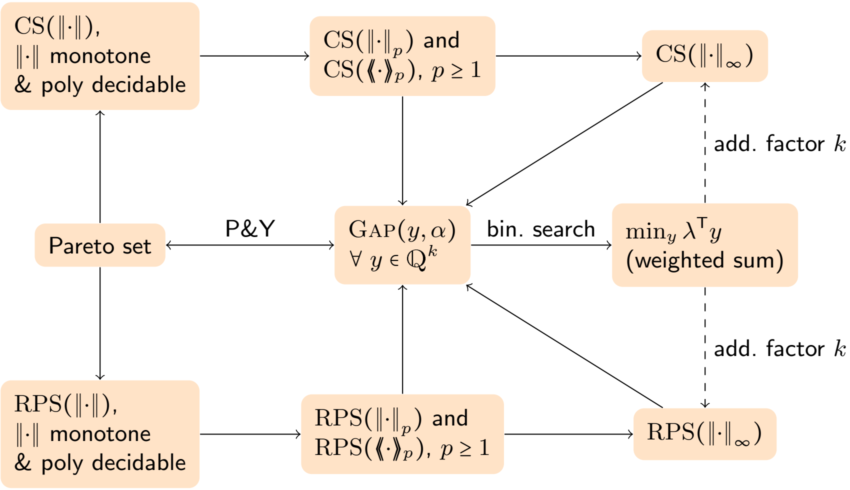

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Extend approximation algorithms based on LP-rounding (e.g. Set Cover, Scheduling) to compromise programming

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$$\Rightarrow \quad r(C\mathcal{R}(x)) \leq \alpha \cdot r(Cx)$$

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- exact solution by dynamic programming in pseudopolynomial time
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- scale and round the instance to achieve polynomial running time
- $x^*, \overline{x}^*$ compromise solutions for original and rounded instance.

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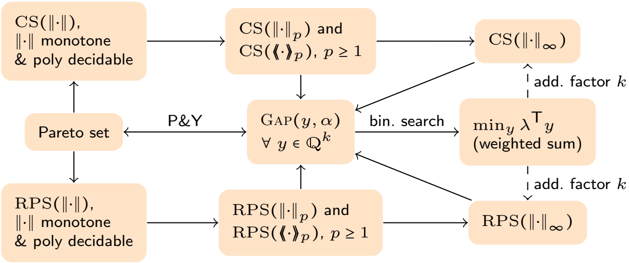
$$\langle\!\langle C\overline{x}^* - y^{\mathrm{id}} \rangle\!\rangle_p + \langle\!\langle y^{\mathrm{id}} \rangle\!\rangle_p \leq (1+\varepsilon) \cdot \left(\langle\!\langle Cx^* - y^{\mathrm{id}} \rangle\!\rangle_p + \langle\!\langle y^{\mathrm{id}} \rangle\!\rangle_p \right)$$

Summary



Approximation Algorithms:

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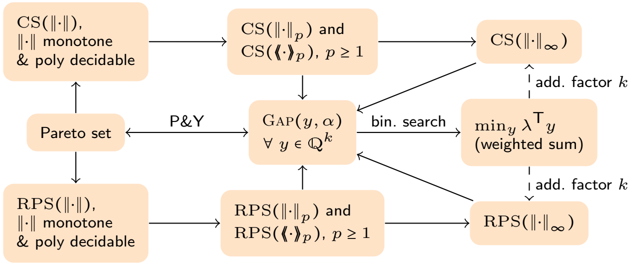
Summary



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