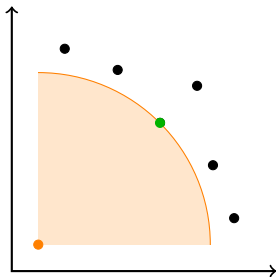


The Power of Compromise

Approximation in Multicriteria Optimization

C. Büsing, **Kai-Simon Goetzmann**, J. Matuschke and S. Stiller



FRICO, August 15, 2012

Multicriteria Optimization

The screenshot displays a route planning application interface. On the left, there is a control panel with the following elements:

- Buttons: "Route berechnen", "Meine Orte", and a print icon.
- Mode selection: Car (selected) and Pedestrian.
- Criteria input:
 - A** Straße des 17. Juni/B2/B5
 - B** Nottingham Jubilee Campus, Stop RA63, UK
- Buttons: "Ziel hinzufügen - Optionen anzeigen" and "ROUTE BERECHNEN".
- Section: "Vorgeschlagene Routen" (Suggested Routes)
- Route A2: 1.305 km, 13 Stunden 6 Minuten

The map on the right shows a route starting in London, UK, and ending in Berlin, Germany. The route is highlighted in blue and passes through several cities including Birmingham, Nottingham, and Manchester. The map also shows major cities in the UK, Europe, and the North Sea region.

Multicriteria Optimization

The screenshot shows a route planning interface. On the left, there are buttons for "Route berechnen" and "Meine Orte". Below these are icons for a car and a pedestrian. Two stops are listed: A: "Straße des 17. Juni/B2/B5" and B: "Nottingham Jubilee Campus, Stop RA63, UK". A blue button "ROUTE BERECHNEN" is visible. Below the input fields, a section titled "Vorgeschlagene Routen" shows a single route: A2, 1.305 km, 13 Stunden 6 Minuten. On the right, a map shows a blue route starting in London, going to Nottingham, then to the first stop in Berlin, and finally to the second stop in Berlin. The map covers parts of the United Kingdom, the Netherlands, Belgium, and Germany.



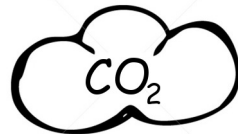
Multicriteria Optimization

The screenshot shows a route planning interface. On the left, there are controls for route calculation, including buttons for "Route berechnen", "Meine Orte", and a "ROUTE BERECHNEN" button. Two starting points are listed: A (StraÙe des 17. Juni/B2/B5) and B (Nottingham Jubilee Campus, Stop RA63, UK). Below, a "Vorgeschlagene Routen" section shows route A2 with a distance of 1.305 km and a duration of 13 hours and 6 minutes. On the right, a map displays a blue route starting from London, passing through Birmingham, Nottingham, Sheffield, and Manchester in the UK, then crossing into Europe through Belgium, the Netherlands, Germany, and ending in Berlin. Green circles labeled A and B mark the starting and ending points on the map.



Multicriteria Optimization

The screenshot shows a route planning interface. On the left, there are buttons for "Route berechnen" and "Meine Orte". Below, there are input fields for two criteria: **A** "Straße des 17. Juni/B2/B5" and **B** "Nottingham Jubilee Campus, Stop RA63, UK". A "ROUTE BERECHNEN" button is present. Below the input fields, a section titled "Vorgeschlagene Routen" shows a route labeled "A2" with a distance of "1.305 km, 13 Stunden 6 Minuten". On the right, a map shows a blue route starting in the UK and ending in Germany, with points A and B marked on the route.



Multicriteria Optimization

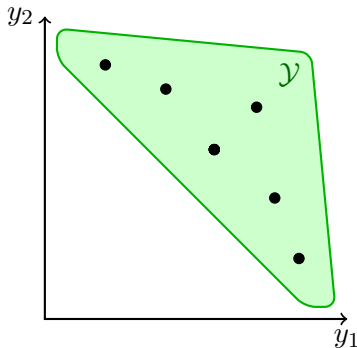
The screenshot shows a route planning interface. On the left, there are controls for route calculation and input fields for stops A and B. Stop A is 'Straße des 17. Juni/B2/B5' and Stop B is 'Nottingham Jubilee Campus, Stop RA63, UK'. A 'ROUTE BERECHNEN' button is visible. Below, a list of suggested routes shows 'A2' with a distance of 1.305 km and a duration of 13 hours and 6 minutes. On the right, a map displays the route from London to Berlin via Nottingham and Birmingham, with stops A and B marked on the map.

$$\min\{y : y \in \mathcal{Y}\} \quad \text{where} \quad \mathcal{Y} \subseteq \mathbb{Z}^k$$

Pareto Optimality

Definition

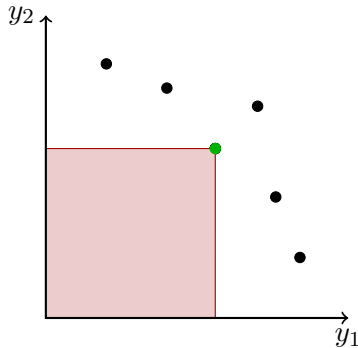
A solution $y \in \mathcal{Y}$ of $\min_{y \in \mathcal{Y}} y$ is *Pareto optimal* if there is no $y' \in \mathcal{Y} \setminus \{y\}$ with $y' \leq y$.



Pareto Optimality

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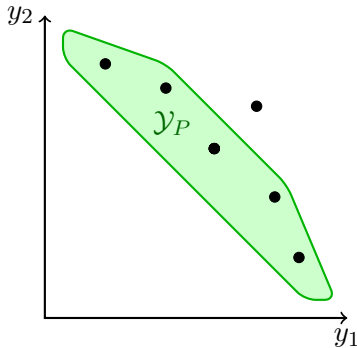
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Reference Point Solutions

Motivation:

- ▶ identify a single, Pareto optimal, balanced solution
- ▶ *reference point methods*:
part of many state-of-the-art MCDM tools,
little theoretical background
- ▶ *The Power of Compromise*:
all Pareto optimal solutions can be CS,
approximation of CS yields approximate Pareto set

1 Introduction

2 Definitions and Notations

3 Approximation

1 Introduction

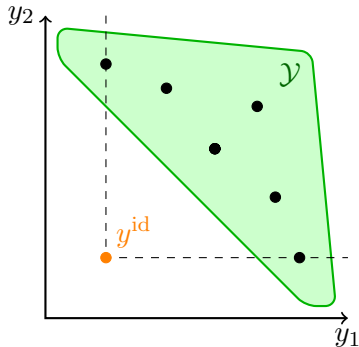
2 Definitions and Notations

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Definition (Ideal Point)

Given a multicriteria optimization problem $\min_{y \in \mathcal{Y}} y$, the *ideal point* $y^{\text{id}} = (y_1^{\text{id}}, \dots, y_k^{\text{id}})$ is defined by

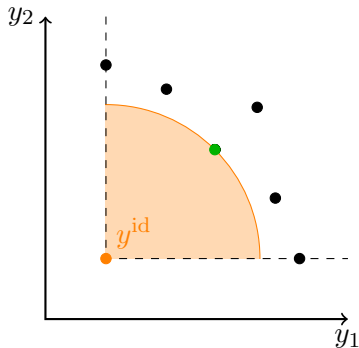
$$y_i^{\text{id}} = \min_{y \in \mathcal{Y}} y_i \quad \forall i.$$



Definition (Compromise Solution, Yu 1973)

Given a multicriteria optimization problem $\min_{y \in \mathcal{Y}} y$ with the ideal point $y^{\text{id}} \in \mathbb{Q}^k$, and a norm $\|\cdot\|$ on \mathbb{R}^k , the *compromise solution* w.r.t. $\|\cdot\|$ is

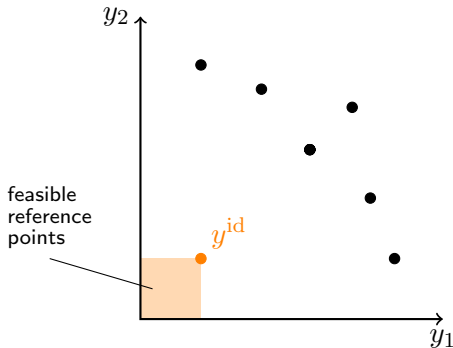
$$y^{\text{cs}} = \min_{y \in \mathcal{Y}} \|y - y^{\text{id}}\|.$$



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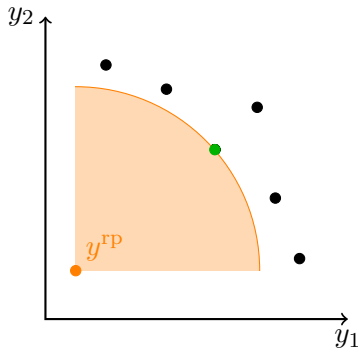
$$y^{\text{cs}} = \min_{y \in \mathcal{Y}} \|y - y^{\text{id}}\|.$$



Definition (Reference Point Solution)

Given a multicriteria optimization problem $\min_{y \in \mathcal{Y}} y$, a **feasible reference point** $y^{\text{rp}} \in \mathbb{Q}^k$, and a norm $\|\cdot\|$ on \mathbb{R}^k , the **reference point solution** w.r.t. $\|\cdot\|$ is

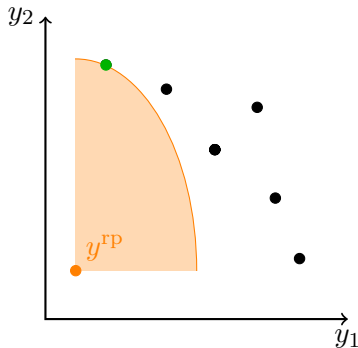
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Definition (Reference Point Solution)

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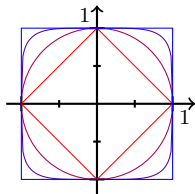


The norms we consider:

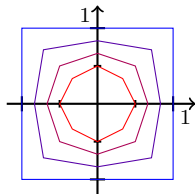
$$\|y\|_p := \left(\sum_{i=1}^k y_i^p \right)^{1/p}, \quad p \in [1, \infty) \quad (\ell^p\text{-Norm})$$

$$\|y\|_\infty := \max_{i=1, \dots, k} y_i \quad (\text{Maximum } (\ell^\infty\text{-Norm)})$$

$$\langle\langle y \rangle\rangle_p := \|y\|_\infty + \frac{1}{p} \|y\|_1, \quad p \in [1, \infty] \quad (\text{Cornered } p\text{-Norm})$$



ℓ^p -Norm



Cornered p -Norm

$p = 1$
 $p = 2$
 $p = 5$
 $p = \infty$

Degree of balancing controlled by adjusting p .

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Weighted version: For any norm and $\lambda \in \mathbb{Q}^k$, $\lambda \geq 0$, $\lambda \neq 0$:

$$\|y\|^\lambda = \|(\lambda_1 y_1, \lambda_2 y_2, \dots, \lambda_k y_k)\|.$$

General properties: Norms we consider are

- ▶ *monotone* (if $y \geq y'$ then $\|y\| \geq \|y'\|$)
- ▶ *polynomially decidable* ($\|y\| \geq \|y'\|$ can be decided in polynomial time)

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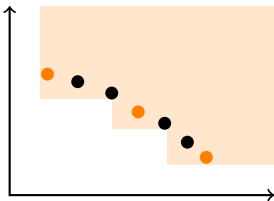
Approximate Pareto sets

Definition (α -approximate Pareto set)

Let \mathcal{Y}_P be the Pareto set of $\min_{y \in \mathcal{Y}} y$, and let $\alpha > 1$.

$\mathcal{Y}_\alpha \subseteq \mathcal{Y}$ is an α -approximate Pareto set if for all $y \in \mathcal{Y}_P$ there is $y' \in \mathcal{Y}_\alpha$ such that

$$y'_i \leq \alpha y_i \quad \forall i = 1, \dots, k$$



How to find approximate Pareto sets

Theorem (Papadimitriou&Yannakakis,2000)

$\text{GAP}(y, \alpha)$ tractable for all $y \in \mathbb{Q}^k$
 $\Rightarrow \alpha^2$ -approximation for the Pareto set.

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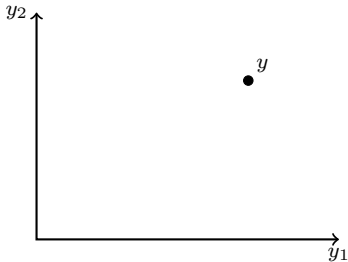
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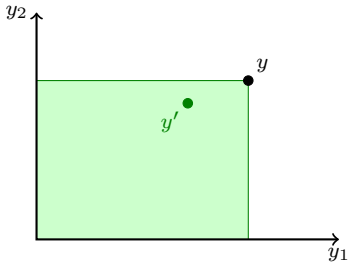
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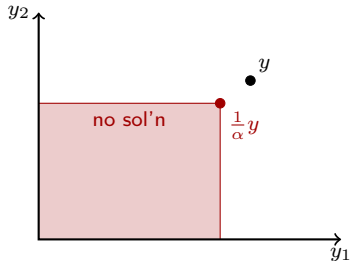
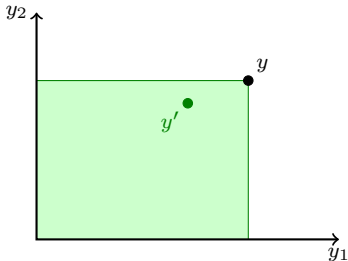
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Approximate Pareto sets \Leftrightarrow approximate CS

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Relate objective value to size of the vectors:

$$\min_{y \in \mathcal{Y}} \underbrace{\|y - y^{\text{id}}\| + \|y^{\text{id}}\|}_{r(y)}$$

We call $r(y)$ the *relative distance*.

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α -approximation of the Pareto set

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\Rightarrow $\text{GAP}(y, \beta)$ tractable for all $y \in \mathbb{Q}^k$, $\beta \in \Theta(\alpha)$.

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Equivalences of approximability

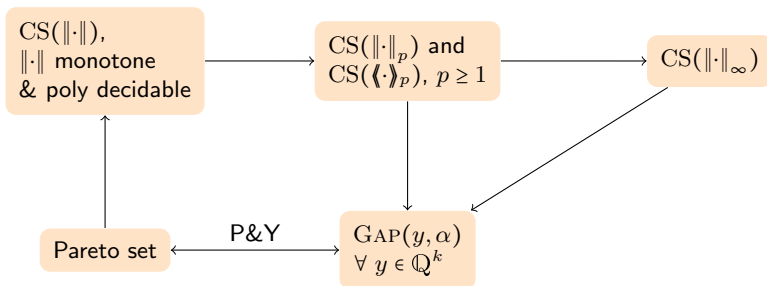
$CS(\|\cdot\|)$,
 $\|\cdot\|$ monotone
& poly decidable

$CS(\|\cdot\|_p)$ and
 $CS(\langle\cdot\rangle_p)$, $p \geq 1$

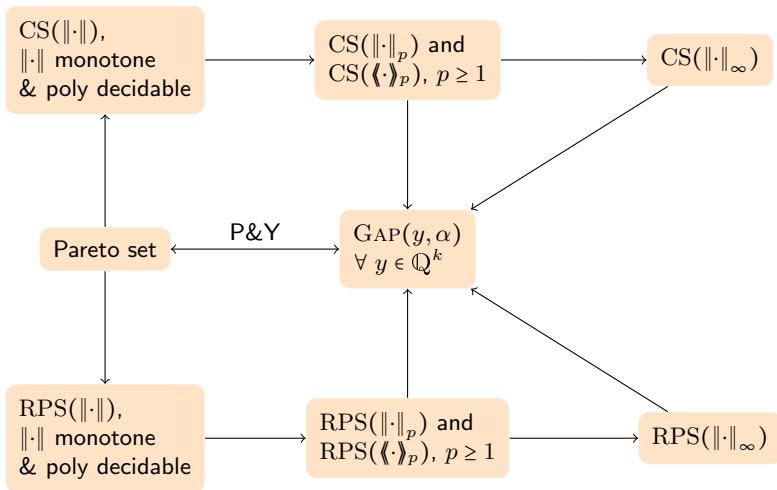
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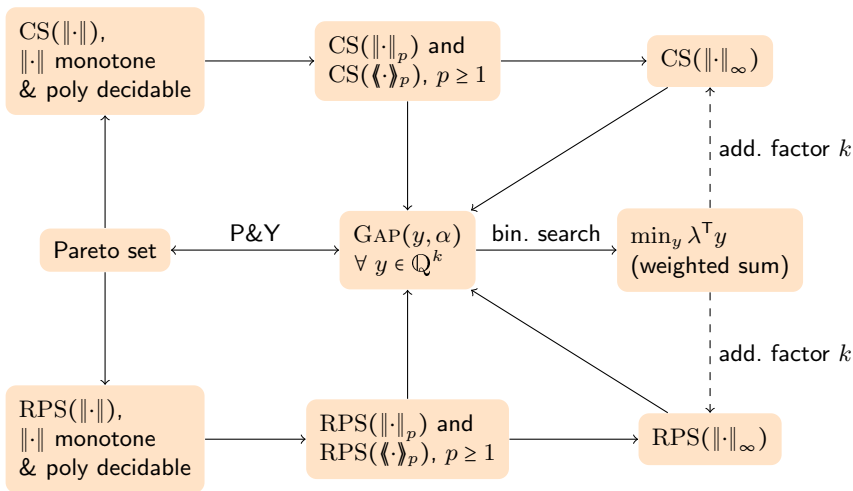
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Approximation through LP-rounding

Extend approximation algorithms based on LP-rounding
(e.g. Set Cover, Scheduling) to compromise programming

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- ▶ LP-relaxation:

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$$Cx - y^{\text{id}} \leq \Delta \cdot \mathbb{1}$$

$$\Delta \geq 0$$

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- ▶ rounding procedure $\mathcal{R} : \mathbb{Q}_{\geq 0}^n \rightarrow \mathbb{N}^n$ with

$$c^\top \mathcal{R}(x) \leq \alpha c^\top x$$

$$\Rightarrow \quad r(C\mathcal{R}(x)) \leq \alpha \cdot r(Cx)$$

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Extend result from min-max-regret robustness (Aissi et al. 2006)
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- ▶ scale and round the instance to achieve polynomial running time
- ▶ x^* , \bar{x}^* compromise solutions for original and rounded instance.

$$\|C\bar{x}^* - y^{\text{id}}\|_{\infty} \leq (1 + \varepsilon) \cdot \|Cx^* - y^{\text{id}}\|_{\infty}$$

Approximation through dynamic programming

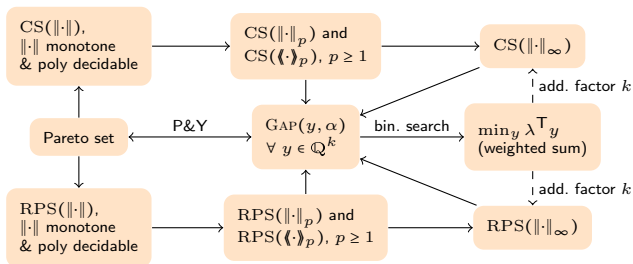
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$$\langle\langle C\bar{x}^* - y^{\text{id}} \rangle\rangle_p + \langle\langle y^{\text{id}} \rangle\rangle_p \leq (1 + \varepsilon) \cdot \left(\langle\langle Cx^* - y^{\text{id}} \rangle\rangle_p + \langle\langle y^{\text{id}} \rangle\rangle_p \right)$$

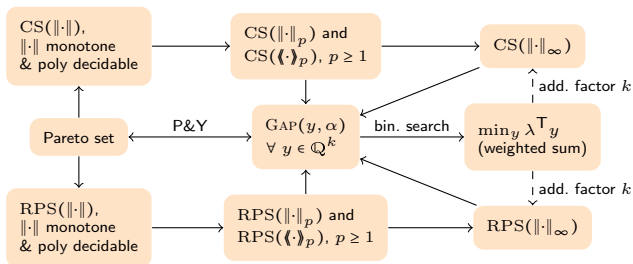
Summary



Approximation Algorithms:

- ▶ Approximations based on LP-relaxations
- ▶ FPTAS based on dynamic programming

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Thank you for your attention.