ADM III: Online Optimization, WS 2013/2014
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Problem Sheet 5

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Exercise 1. We consider deterministic algorithms for the online bipartite matching problem as discussed in class. We are given the vertex set L of a bipartite graph $G = (L \cup R, E)$, and the vertices in R are revealed online one by one. When a vertex $j \in R$ arrives then an online algorithm has to make an irrevocable decision on matching j to any of its neighbors in L. The goal is to maximize the cardinality of the matching (at any step).

Prove the following theorems.

(a) **Theorem.** No deterministic online algorithm can obtain a competitive ratio strictly less than 2.

(b) Consider the simple greedy algorithm: Match a new vertex j to an arbitrary unmatched neighbor in L, if there is any.

Theorem. The greedy algorithm is 2-competitive.

Hints: It may be useful to prove the following two lemmas.

Lemma. The greedy algorithm computes a maximal matching.

Lemma. Let G be any graph, M^* a maximum matching, and M a maximal matching. Then $|M^*| \leq 2|M|$.

Exercise 2. Consider the offline vertex cover problem. Given is a graph $G = (V, E)$ with non-negative weights $w_v, v \in V$. Find a vertex cover of minimum total cost, i.e., a subset $V' \subseteq V$ such that each edge $e \in E$ has an end point in V' and $\sum_{v \in V'} w_v$ is minimized.

Formulate an LP relaxation and its dual. Consider the following primal dual algorithm.

1. Initialize the primal and dual variables with 0.
2. While there is an uncovered edge:
   (a) Pick an uncovered edge, and increase the dual variable until some constraint becomes tight.
   (b) Set all primal constraints corresponding to the tight dual constraints to 1.

Show that this algorithm constructs a feasible vertex cover of total weight at most twice value of an optimal vertex cover.

Exercise 3. We consider an online variant of the classical knapsack problem with a knapsack of capacity 1. The items $w_i \in (0, 1]$ arrive online, one after the other, and the online algorithm has to decide immediately whether to pack each item. The objective is to maximize $\sum_{i \in K} w_i \leq 1$, where K is the set of items packed into the knapsack.

(a) Show that no deterministic algorithm has a bounded competitive ratio for this problem.

(b) Consider the offline algorithm GREEDY that first sorts the items in *any* order $w_1, w_2, \ldots$, and then finds the smallest j, such that $\sum_{i \leq j} w_i > 1$. If $\sum_{i < j} w_i > w_j$, GREEDY packs $\{w_1, \ldots, w_{j-1}\}$, else GREEDY packs $\{w_j\}$. Show that GREEDY $\geq \frac{1}{2}$OPT.

(c) Design a randomized 2-competitive algorithm.