Characterization of Polytopes

Lemma 14.9.

Let $P = \{x \mid Ax \leq b\}$ and $v \in P$. Then v is a vertex of $P \iff v$ cannot be written as a convex combination of vectors in $P \setminus v$.

Proof: ... □

Theorem 14.10.

A polytope is equal to the convex hull of its vertices.

Proof: ... □

Theorem 14.11.

A set P is a polytope $\iff$ there exists a finite set V such that V is the convex hull of V.

Proof: ... □

A Guideline to Min-Max Relations

Min-max relationships are often the key that lead to combinatorial algorithms for discrete optimization problems.

Theorem 14.11 provides a general method for obtaining such min-max relations:

1. Represent the combinatorial problem as an optimization problem over a finite set S of vectors (e.g., the characteristic vectors of matchings.)
2. Find a linear description of the convex hull of S.
3. Apply the Duality Theorem of Linear Programming and Theorem 14.2 to obtain a min-max relation for the combinatorial problem.

Note: Theorem 14.11 only states that step 2 is possible. Finding such a linear description is the "art of polyhedral combinatorics".

Integral Polytopes

In the following, we restrict to rational polyhedra, i.e., polyhedra defined by rational linear systems.

Note that this restriction is not really harmful since we deal almost exclusively with integral objects.

Definition 14.12.

A rational polyhedron is integral if every nonempty face contains an integral vector.

Since it suffices to consider minimal faces in the Definition and by Observation 14.8 we observe:

Observation 14.13.

A pointed rational polyhedron is integral $\iff$ all of its vertices are integral.

Hoffman's Characterization of Integral Polytopes

Proving that a polyhedron is integral is an important (but sometimes difficult) task in combinatorial optimization.

For example, the celebrated MAX-FLOW-MIN-CUT Theorem follows directly from the integrality of the polyhedron

$$\{y \in \mathbb{R}^E_+ \mid y(P) \geq 1 \text{ for each } s,t\text{-path } P \text{ in } G\}.$$

Theorem 14.14 (Hoffman 1974).

A rational polytope P is integral $\iff$ for all integral vectors w the optimal value of $\max\{w^T x \mid x \in P\}$ is an integer.

Proof: ... □

Note: The result can be extended to unbounded polyhedra.