Announcements

- Possible dates for oral exam: July 30 & 31, September 24–26
- Next week: only tutorial sessions
- In particular, no class on Tuesday, July 3, and Friday, July 6; no exercise session on Thursday, July 5.
- Next semester:
  - ADM III – Approximation Algorithms
  - Seminar: Advanced topics in Combinatorial Optimization
    More information on Thursday, July 26, 16:00, in room MA 517

Weighted Matroid Intersection Problem

Given: Two matroids $(E, \mathcal{F}_1)$, $(E, \mathcal{F}_2)$ on a common finite set E, $c \in \mathbb{R}^E$.

Task: Find $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ maximizing $c(X)$.

Examples

- Maximum weight matching problem on bipartite graphs.
- Given a digraph D with weights on the arcs, find a branching (forest in which every node has indegree at most one) of maximum weight.

In the following, we restrict to the unit-weight case, i. e., the problem of finding $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ maximizing $|X|$.

We refer to this problem as the Matroid Intersection Problem.

Matroid Intersection: Preliminaries

Notation: Let $(E, \mathcal{F})$ be a matroid. For $X \in \mathcal{F}$ and $f \in E$ let

$$C(X,f) := \begin{cases} \emptyset & \text{if } X \cup \{f\} \in \mathcal{F}, \\ \text{unique circuit in } X \cup \{f\} & \text{otherwise.} \end{cases}$$

Lemma 16.13.

Let $(E, \mathcal{F})$ be a matroid, $X \in \mathcal{F}$, $e_1, \ldots, e_s \in X$, and $f_1, \ldots, f_s \notin X$ with

- i $e_k \in C(X, f_k)$ for $k = 1, \ldots, s$ and
- ii $e_j \notin C(X, f_k)$ for $j < k$.

Then, $(X \setminus \{e_1, \ldots, e_s\}) \cup \{f_1, \ldots, f_s\} \in \mathcal{F}$.

Proof:. . . □

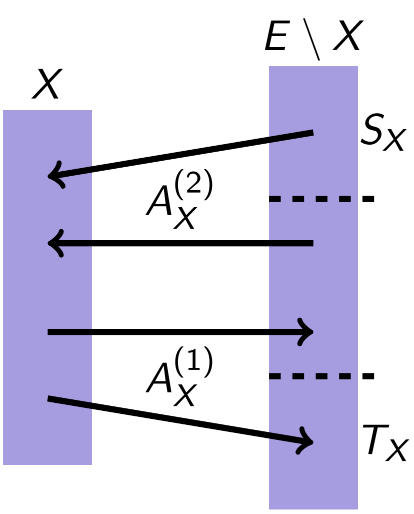
Matroid Intersection: Basic Structures for Algorithm

Given $X \in \mathcal{F}_1 \cap \mathcal{F}_2$, let

$$S_X := \{f \in E \setminus X \mid X \cup \{f\} \in \mathcal{F}_1\}$$
$$T_X := \{f \in E \setminus X \mid X \cup \{f\} \in \mathcal{F}_2\}$$

If $f \in S_X \cap T_X$, then $X \cup \{f\} \in \mathcal{F}_1 \cap \mathcal{F}_2$.

Otherwise, if $S_X \cap T_X = \emptyset$, then construct a digraph D_X as follows:



$$A_X^{(1)} := \{(e,f) \mid f \in E \setminus X,\ e \in C_1(X,f) \setminus \{f\}\}$$

$$A_X^{(2)} := \{(f,e) \mid f \in E \setminus X,\ e \in C_2(X,f) \setminus \{f\}\}$$

$$D_X := (E, A_X^{(1)} \cup A_X^{(2)})$$

Matroid Intersection: Basic Results for Algorithm

The following two lemmas are the key to Edmond's Algorithm for solving the matroid intersection problem.

Lemma 16.14.

Let $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ and $f_0, e_1, f_1, \ldots, e_s, f_s$ the nodes of a shortest f_0-f_s-path in D_X with $f_0 \in S_X$ and $f_s \in T_X$. Then,

$$X' := \big(X \setminus \{e_1, \ldots, e_s\}\big) \cup \{f_0, f_1, \ldots, f_s\} \in \mathcal{F}_1 \cap \mathcal{F}_2 \ .$$

Notice that $|X'| = |X| + 1$.

Proof:. . . □

Lemma 16.15.

A subset $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ has maximum cardinality if and only if there is no S_X-T_X-path in D_X.

Proof:. . . □

Matroid Intersection Theorem and Algorithm

Theorem 16.16 (Matroid Intersection Theorem of Edmonds 1970).

For matroids $(E, \mathcal{F}_1)$, $(E, \mathcal{F}_2)$

$$\max_{X \in \mathcal{F}_1 \cap \mathcal{F}_2} |X| = \min_{Q \subseteq E} \big(r_1(Q) + r_2(E \setminus Q)\big) \ .$$

□

Edmond's Matroid Intersection Algorithm

1. let $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ (e. g., $X := \emptyset$);
2. while there is an S_X-T_X-path in D_X
3. apply Lemma 16.14 and increase $|X|$ by one;

Theorem 16.17.

The matroid intersection problem can be solved in $O(|E|^3)$ time.

Proof: D_X can be constructed with $O(|E|^2)$ independence tests. □

Intersection of Three or More Matroids

Theorem 16.18.

Given three matroids $(E, \mathcal{F}_1)$, $(E, \mathcal{F}_2)$, $(E, \mathcal{F}_3)$ (by oracles), and a number $k \in \mathbb{N}$, it is NP-complete to decide whether there is $F \in \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$ with $|F| \geq k$.

Proof: Reduction of the Directed Hamiltonian Path Problem. □