Announcements

- Possible dates for oral exam: July 30 & 31, September 24–26
- Next week: only tutorial sessions
- In particular, no class on Tuesday, July 3, and Friday, July 6; no exercise session on Thursday, July 5.
- Next semester:
  - ADM III – Approximation Algorithms
  - Seminar: Advanced topics in Combinatorial Optimization

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Chapter 16: Matroids

(cp. Cook, Cunningham, Pulleyblank & Schrijver, Chapter 8)

Greedy Algorithm

Reminder: Kruskal's algorithm finds a forest of max. weight in $G = (V, E)$.

Let $\mathcal{F} := \{F \subseteq E \mid (V, F) \text{ forest}\}$ denote the set of all forests in G.

Note: For every weight function $c \in \mathbb{R}^E$, the following Greedy Algorithm finds a forest F of maximum weight $c(F)$.

Greedy Algorithm:

1. Set $F := \emptyset$;
2. While $\exists e \in E \setminus F$ with $c_e > 0$ and $F \cup \{e\} \in \mathcal{F}$
3. Choose such e with c_e maximum;
4. Replace F by $F \cup \{e\}$;

Q: Can we apply the same algorithm to other discrete structures $\mathcal{F} \subseteq 2^E$ (e. g., matchings)?

Examples: ...

Independence Systems

Definition 16.1.

Given a finite set E and $\mathcal{F} \subseteq 2^E$, the set system $(E, \mathcal{F})$ is an independence system if

(i) $\emptyset \in \mathcal{F}$, and

(ii) $X \subseteq Y$, $Y \in \mathcal{F} \implies X \in \mathcal{F}$

A set $F \in \mathcal{F}$ is called independent, a set $G \in 2^E \setminus \mathcal{F}$ is called dependent. The minimal dependent sets are called circuits, a maximal independent set is called basis.

Maximization Problem for Independence Systems
Given: Independence system $(E, \mathcal{F})$ and $c \in \mathbb{R}^E$.
Task: Find $F \in \mathcal{F}$ with $c(F) := \sum_{e \in F} c_e$ maximum.

Minimization Problem for Independence Systems
Given: Independence system $(E, \mathcal{F})$ and $c \in \mathbb{R}^E$.
Task: Find a basis $B \in \mathcal{F}$ with $c(B)$ minimum.

Optimization Problems on Independence Systems

- Maximum Stable Set Problem

$$E := \text{set of nodes} \qquad \mathcal{F} := \{X \subseteq E \mid X \text{ stable}\}$$

- Knapsack Problem
- Minimum Spanning Tree Problem
- Maximum Weight Forest Problem
- Maximum Weight Matching Problem
- Shortest Path Problem
- Traveling Salesperson Problem

Matroids

- For $\mathcal{F} \subseteq 2^E$, the cardinality $|\mathcal{F}|$ is in general exponential in $|E|$.
- We thus assume that $\mathcal{F}$ is given by an oracle that answers the following question: Given $F \subseteq E$, is $F \in \mathcal{F}$?

Definition 16.2.

An independence system $(E, \mathcal{F})$ is a matroid if

$$X, Y \in \mathcal{F},\ |X| > |Y| \quad \Longrightarrow \quad \exists e \in X \setminus Y \text{ with } Y \cup \{e\} \in \mathcal{F} \ .$$

Examples:

- Consider a matrix $A \in \mathbb{F}^{d \times n}$ over some field $\mathbb{F}$. Let E be the set of columns of A and $\mathcal{F} \subseteq 2^E$ the family of linear independent subsets. (linear matroid, vector matroid, matroid representable over field $\mathbb{F}$)
- Consider a graph $G = (V, E)$ and let $\mathcal{F} \subseteq 2^E$ be the family of forests in G. (cycle matroid, graphic matroid)
- Let E be a finite set, $k \in \mathbb{N}$, and $\mathcal{F} := \{X \subseteq E \,\big|\, |X| \leq k\}$. (uniform matroid)

Characterization of Matroids

Definition 16.3.

Let $(E, \mathcal{F})$ be an independence system. For $X \subseteq E$, a maximal (w.r.t. inclusion) independent subset of X is a basis of X.

Theorem 16.4.

Let $(E, \mathcal{F})$ be an independence system. The following statements are equivalent:

1. $(E, \mathcal{F})$ is a matroid.
2. $X, Y \in \mathcal{F}$, $|X| = |Y| + 1 \Longrightarrow \exists e \in X \setminus Y$ with $Y \cup \{e\} \in \mathcal{F}$.
3. For all $X \subseteq E$, all bases of X have the same cardinality.

Proof:. . . □

Algorithmic Characterization of Matroids

Theorem 16.5 (Rado '57).

Let $(E, \mathcal{F})$ be an independence system. The Greedy Algorithm finds a max-weight independent set for *every* $c \in \mathbb{R}^E$ if and only if $(E, \mathcal{F})$ is a matroid.

Proof: Will be given later, in more general context (Theorem 16.11). □

Rank and Rank Quotient of Independence Systems

Definition 16.6.

Let $(E, \mathcal{F})$ be an independence system and $X \subseteq E$.

i. The rank $r(X)$ of X is the maximum cardinality of a basis of X:

$$r(X) := \max\{|Y| \mid Y \subseteq X,\ Y \in \mathcal{F}\}$$

ii. The lower rank $\rho(X)$ of X is the minimum cardinality of a basis of X:

$$\rho(X) := \min\{|Y| \mid Y \text{ basis of } X\}$$

iii. The rank quotient of $(E, \mathcal{F})$ is

$$q(E, \mathcal{F}) := \min_{X \subseteq E} \frac{\rho(X)}{r(X)} \leq 1 \ .$$

Observation 16.7.

Independence system $(E, \mathcal{F})$ is a matroid if and only if $q(E, \mathcal{F}) = 1$. □

Characterization of Matroids via Bases

Theorem 16.8.

Let E be a finite set and $\mathcal{B} \subseteq 2^E$. Then, $\mathcal{B}$ is the set of bases of some matroid $(E, \mathcal{F})$ if and only if

i. $\mathcal{B} \neq \emptyset$;

ii. $B_1, B_2 \in \mathcal{B},\ e \in B_1 \setminus B_2 \implies \exists f \in B_2 \setminus B_1$ with $(B_1 \setminus \{e\}) \cup \{f\} \in \mathcal{B}$.

Proof: Exercise. □

Characterization of Matroids via Rank Functions

Theorem 16.9.

Let E be a finite set and $r : 2^E \to \mathbb{Z}_{\geq 0}$. Then, the following statements are equivalent:

i r is the rank function of some matroid $(E, \mathcal{F})$.

ii For all $X, Y \subseteq E$:

- a $r(X) \leq |X|$
- b $X \subseteq Y \implies r(X) \leq r(Y)$
- c $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$ (i. e., r is submodular)

iii For all $X, Y \subseteq E$ and $e, f \in E$:

- a $r(\emptyset) = 0$
- b $r(X) \leq r(X \cup \{e\}) \leq r(X) + 1$
- c $r(X) = r(X \cup \{e\}) = r(X \cup \{f\}) \implies r(X \cup \{e, f\}) = r(X)$

□

Characterization of Matroids via Circuits

Theorem 16.10.

Let E be finite set. Then, $\mathcal{C} \subseteq 2^E$ is the set of circuits of an independence system $(E, \mathcal{F})$ with $\mathcal{F} = \{F \subseteq E \mid \nexists C \in \mathcal{C} : C \subseteq F\}$ if and only if

1 $\emptyset \notin \mathcal{C}$

2 For all $C_1, C_2 \in \mathcal{C}$: $C_1 \subseteq C_2 \implies C_1 = C_2$

In this case, the following statements are equivalent:

i $(E, \mathcal{F})$ is a matroid

ii $X \in \mathcal{F}$, $e \in E \implies X \cup \{e\}$ contains at most one circuit

iii $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$, $e \in C_1 \cap C_2 \implies \exists C_3 \in \mathcal{C}$: $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$

iv $C_1, C_2 \in \mathcal{C}$, $e \in C_1 \cap C_2$, $f \in C_1 \setminus C_2$
$\implies \exists C_3 \in \mathcal{C}$ with $f \in C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$

□

Greedy Algorithm for Independence Systems

Input: independence system $(E, \mathcal{F})$ given by oracle, $c \in \mathbb{R}^E$
Output: $F \in \mathcal{F}$

1. sort $E = \{e_1, e_2, \ldots, e_n\}$ such that $c(e_1) \geq c(e_2) \geq \cdots \geq c(e_n)$;
2. set $F := \emptyset$;
3. for $i = 1$ to n do: if $F \cup \{e_i\} \in \mathcal{F}$, then set $F := F \cup \{e_i\}$;

Theorem 16.11.

Let $(E, \mathcal{F})$ be an independence system and $c \in \mathbb{R}^E$. Let $G(E, \mathcal{F}, c)$ be the cost of the solution found by the Greedy Algorithm. Then,

$$q(E, \mathcal{F}) \leq \frac{G(E, \mathcal{F}, c)}{\mathrm{OPT}(E, \mathcal{F}, c)} \leq 1 \ .$$

There is a cost function $c \in \mathbb{R}^E$ for which the lower bound is attained.

Proof:... □

Polyhedral Description of Matroids

Theorem 16.12.

Let $(E, \mathcal{F})$ be a matroid with rank function $r : 2^E \to \mathbb{Z}_{\geq 0}$. The convex hull of the incidence vectors of all elements of $\mathcal{F}$ is equal to

$$\left\{ x \in \mathbb{R}^E \,\middle|\, x \geq 0, \sum_{e \in X} x_e \leq r(X) \text{ for all } X \subseteq E \right\} \ .$$

Proof:... □