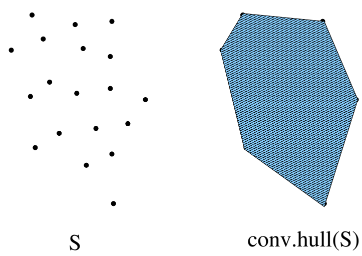
Chapter 14:
Integer Linear Programming

(cp. Cook, Cunningham, Pulleyblank & Schrijver, Chapter 6)

Convex Hulls

Let $S \subseteq \mathbb{R}^n$ be a finite (maybe exponentially large) set.



Given $w \in \mathbb{R}^n$, we consider discrete optimization problems of type:

$$\max\{w^T x \mid x \in S\}.$$

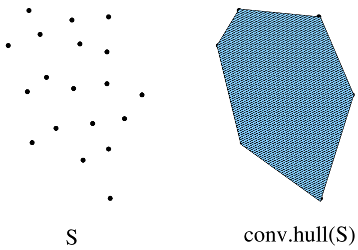
Definition 14.1.

The convex hull of a finite set S (denoted by conv.hull(S)) is the set of all vectors that can be written as convex combinations of S.

Convex Hulls

Theorem 14.2.

$\max\{w^T x \mid x \in S\} = \max\{w^T x \mid x \in \text{conv.hull}(S)\}$.



Proof: . . . □

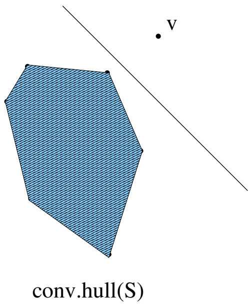
We have now replaced a finite optimization problem by an infinite one. Is this helpful?

Yes! Since we may use the nice geometric properties of conv.hull(S)!

Separating Hyperplane

Theorem 14.3.

Let $S \subseteq \mathbb{R}^n$ be a finite set and let $v \in \mathbb{R}^n \setminus \text{conv.hull}(S)$. Then there exists an inequality $w^T x \leq t$ that separates v from conv.hull(S), i.e., $w^T s \leq t$ for all $s \in \text{conv.hull}(S)$ but $w^T v > t$.



Proof: . . . □

Example: Matching Polyhedra

Given graph $G = (V, E)$ let $\mathcal{PM}(G) \subseteq \mathbb{R}^E$ denote the set of characteristic vectors its perfect matchings.

Recall: Let $w \in \mathbb{R}^E$ and suppose $\mathcal{PM}(G) \neq \emptyset$. Then the Blossom algorithm determines a perfect matching $\bar{x}$ that minimizes $w^T x$ over all $x \in \mathcal{PM}(G)$. Moreover, the dual solution proves that $\bar{x}$ minimizes $w^T x$ even over all solutions of

$$\begin{aligned} x(\delta(v)) &= 1 && \text{for all } v \in V \\ x(\delta(S)) &\geq 1 && \text{for all } S \subseteq V, |S| \geq 3 \text{ and odd} \\ x_e &\geq 0 && \text{for all } e \in E. \end{aligned}$$

Theorem 14.4 (Perfect Matching Polytope Theorem).
For any graph $G = (V, E)$, the convex hull of $\mathcal{PM}(G)$ is identical to the set of solutions of the linear inequality system above.

Proof: ... □

Some Definitions

- A polyhedron is the solution set of a finite set of linear inequalities.
- A polytope is a bounded polyhedron (i.e., it contains no infinite half-line).
- An inequality $w^T x \leq t$ is valid for a polyhedron P if $P \subseteq \{x \mid w^T x \leq t\}$.
- $H := \{x \mid w^T x = t\}$ is a supporting hyperplane w.r.t. polyhedron P if $w^T x \leq t$ is valid for P and $P \cap H \neq \emptyset$.
- The intersection of a polyhedron P with one of its supporting hyperplanes is called face of P. (By convention: also P and the empty set are faces. The others are called proper faces.)

Faces of a Polyhedron

Theorem 14.5.

Let $P = \{x \mid Ax \le b\}$. A nonempty set $F \subseteq P$ is a face of P $\iff$ $F = \{x \in P \mid A'x = b'\}$ for some subsystem $A'x \le b'$ of $Ax \le b$.

Proof: . . . □

Thus, a polyhedron has only finitely many distinct faces.

Observation 14.6.

Let F be an inclusion-wise minimal face of $P = \{x \mid Ax \le b\}$. Then $F = \{x \mid A'x = b'\}$ for some subsystem $A'x \le b'$ of $Ax \le b$. Moreover, $\text{rank}(A') = \text{rank}(A)$.

Proof: Exercise □

Vertices

Definition 14.7.

A vector $v \in P$ is a vertex if $\{v\}$ is a face of P. Polyhedron P is called pointed if it contains at least one vertex.

Observation 14.8.

If a polyhedron P is pointed then every minimal nonempty face of P is a vertex.

Proof: . . . □