Postman Problem

Given: Connected graph $G = (V, E)$ with edge weights $c \in \mathbb{R}^E_{\geq 0}$.

Task: Find a postman tour, i. e., a closed path traversing every edge in G at least once. Minimize the total weight of the path.

Remarks:

- Equivalently, instead of minimizing the total weight, one can minimize the cost of the "extra" edge-traversals.
- If there is a postman tour with no extra edge traversals, then that tour is optimal.

Definition 13.33.

A closed edge-simple path P such that $E(P) = E$ is called an Euler tour.

Theorem 13.34.

A connected graph G has an Euler tour if and only if every node of G has even degree. □

Integer Programming Formulation

Let $x_e \in \mathbb{Z}_{\geq 0}$ be the number of extra traversals of edge e for each $e \in E$.

By the last theorem, we can formulate the postman problem as follows:

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e \cdot x_e \\ \text{s.t.} \quad & x\big(\delta(v)\big) \equiv |\delta(v)| \mod 2 \qquad \text{for all } v \in V \\ & x \in \mathbb{Z}^E_{\geq 0} \end{aligned}$$

Remarks:

- There is an optimal solution with $x_e \in \{0, 1\}$ for all $e \in E$.
- Thus, the task is to find $J \subseteq E$ such that

$$|\delta(v) \cap J| \equiv |\delta(v)| \mod 2 \qquad \text{for all } v \in V.$$

We call such a J a postman set.

T-Joins

Definition 13.35.

Let $G = (V, E)$ be a graph, and let $T \subseteq V$ such that $|T|$ is even. A T-join of G is a set J of edges such that

$$|J \cap \delta(v)| \equiv |T \cap \{v\}| \mod 2 \qquad \text{for all } v \in V.$$

That is, the odd-degree nodes of (V, J) are exactly the elements of T.

Optimal T-join problem

Given: Graph $G = (V, E)$ with weights $c \in \mathbb{R}^E$ and $T \subseteq V$ with $|T|$ even.

Task: Find a T-join J of G such that $c(J)$ is minimum.

Examples:

- Postman sets. Let $T := \{v \in V \mid |\delta(v)| \text{ odd}\}$. Then the T-joins are precisely the postman sets.
- s-t-paths. Let $T := \{s, t\} \subseteq V$. Then, every T-join contains the edge set of an s-t-path.

Solving the Optimal T-Join Problem

Lemma 13.36.

Let J' be a T'-join of G. Then J is a T-join of G if and only if $J \triangle J'$ is a $T \triangle T'$-join of G.

Proof:... $\square$

Lemma 13.37.

Every minimal (w.r.t. inclusion) T-join is the union of the edge-sets of $|T|/2$ edge-disjoint simple paths, which join the nodes in T in pairs.

Proof:... $\square$

Lemma 13.38.

Suppose that $c \geq 0$. Then there is an optimal T-join that is the union of $|T|/2$ edge-disjoint shortest paths joining the nodes of T in pairs.

Proof:... $\square$

Solving the Optimal T-Join Problem (cont.)

If $c \geq 0$, then the optimal T-join problem can be solved as a minimum-weight perfect matching problem:

Theorem 13.39.

Let $c \geq 0$. For $u, v \in T$, let d_{uv} the minimum weight of a u-v-path in G. The minimum weight of a T-join equals the minimum weight of a perfect matching in the complete graph with nodes T and edge weights d_{uv}.

Proof:... □

Next consider the optimal T-join problem with arbitrary weights $c \in \mathbb{R}^E$.

Lemma 13.40.

Let $N := \{e \in E \mid c_e < 0\}$ and $T' \subseteq V$ the set of nodes of odd degree in (V, N). Then J is an optimal T-join w.r.t. cost vector c if and only if $J \triangle N$ is an optimal $(T \triangle T')$-join w.r.t. cost vector $|c|$.

Proof:... □

Optimal T-Join Algorithm

1. Let $N := \{e \in E \mid c_e < 0\}$ and $T' \subseteq V$ the set of nodes of odd degree in (V, N). Replace c by $|c|$ and T by $T \triangle T'$.
2. For every pair $u, v \in T$, find least-weight u-v-path P_{uv} w.r.t. c. Let d_{uv} be the weight of P_{uv}.
3. Find a minimum-weight perfect matching M in the complete graph with nodes T and edge weights d_{uv}.
4. Let J be the symmetric difference of the edge-sets of paths P_{uv} for $\{u, v\} \in M$.
5. Replace J by $J \triangle N$.

Remarks:

- If $c \geq 0$, steps 2 to 4 will do the job.
- The running time of the algorithm is bounded by $O(n^4)$.
- The algorithm can be used to solve the shortest s-t-path problem in undirected graphs without negative circuits.

LP Formulation for the Optimal T-Join Problem

Definition 13.41.

A set $S \subseteq V$ is T-odd if $|S \cap T|$ is odd. In this case, $\delta(S)$ is a T-cut.

Observation: The characteristic vector of any T-join of G satisfies

$$x\big(\delta(S)\big) \geq 1 \enspace .$$

Thus, the weight of any T-join is at least equal to the optimum LP value:

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e \cdot x_e \\ \text{s.t.} \quad & x(D) \geq 1 \qquad \text{for all } T\text{-cuts } D \\ & x \geq 0 \end{aligned}$$

Theorem 13.42.

If $c \geq 0$, then the minimum weight of a T-join of G is equal to the optimum LP value.

Proof: See book of Cook et al., proof of Theorem 5.28. □

LP Formulation for Arbitrary Weights

If there is an edge $e \in E$ with $c_e < 0$, the LP above is unbounded.

Consider the following LP relaxation:

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e \cdot x_e \\ \text{s.t.} \quad & x\big(\delta(S) \setminus F\big) - x(F) \geq 1 - |F| \quad \forall\, S \subseteq V,\, F \subseteq \delta(S) : |F| + |S \cap T| \text{ odd} \\ & 0 \leq x \leq 1 \end{aligned}$$

Theorem 13.43.

The minimum cost of a T-join is equal to the optimum LP value.

Proof: See book of Cook et al., proof of Theorem 5.30. □