The Triangle Inequality

Definition 13.27.

Let $G = (V, E)$ be a complete graph and $c \in \mathbb{R}^E_{\geq 0}$. Then c satisfies the triangle inequality if

$$c_{\{u,v\}} + c_{\{v,w\}} \geq c_{\{u,w\}} \qquad \text{for all } u, v, w \in V.$$

Theorem 13.28.

Let $G = (V, E)$ be a complete graph with an even number of nodes and $c \in \mathbb{R}^E_{\geq 0}$. If c satisfies the triangle inequality, then the Blossom Algorithm started with $(y, Y) = 0$ finds an optimal dual solution (y, Y) with $y \geq 0$.

Proof: See book of Cook et al., proof of Theorem 5.20. □

Maximum-Weight Matching Problem

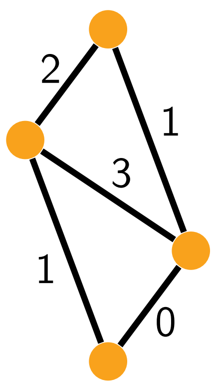
Given: A graph $G = (V, E)$ and $c \in \mathbb{R}^E$.

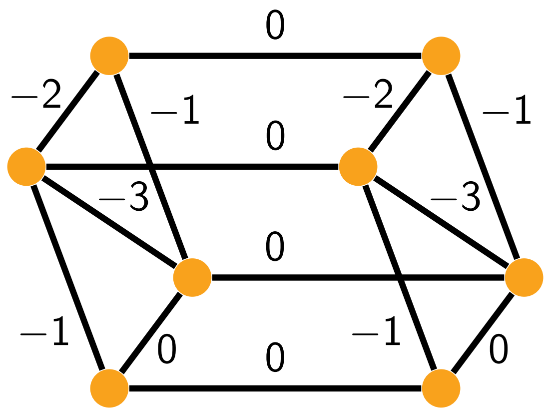
Task: Find a matching M of G such that $c(M)$ is maximum.

Observation: the maximum-weight matching problem can be reduced to the minimum-weight perfect matching problem as follows:

- multiply all edge weights by -1, i. e., replace c with $-c$;
- make a copy G^* of G; give all edges of G^* the same weights as in G;
- join each node in G to its copy in G^* by an edge of weight 0.

Example:





LP Formulation for Maximum-Weight Matching Problem

$$\begin{aligned}
\max \quad & \sum_{e \in E} c_e \cdot x_e \\
\text{s.t.} \quad & x(\delta(v)) \le 1 && \text{for all } v \in V \\
& x(\gamma(S)) \le (|S|-1)/2 && \text{for all } S \subseteq V,\ |S| \ge 3,\ |S| \text{ odd} \\
& x \ge 0
\end{aligned}$$

Theorem 13.29.
The maximum weight of a matching of G is equal to the optimal value of the LP above. □

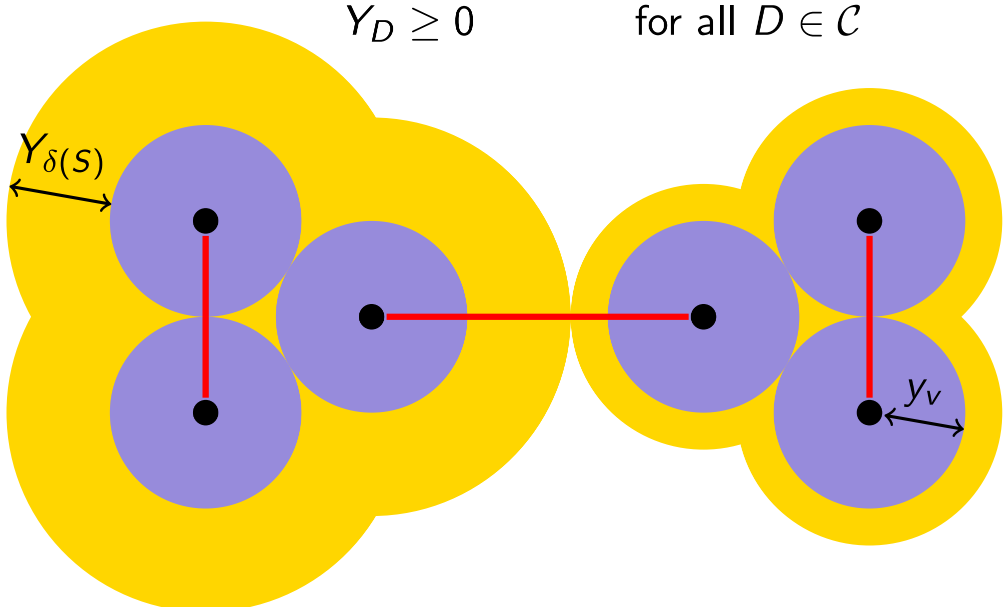
Remarks:

- Theorem can be proved by applying the Matching Polytope Theorem to the graph constructed on the last slide.
- Alternatively, it can be proved directly by a modified version of the Blossom Algorithm. (See book of Cook et al.)

Geometric Interpretation of the Dual LP

Dual perfect matching LP:

$$\begin{aligned}
\max \quad & \sum_{v \in V} y_v + \sum_{D \in \mathcal{C}} Y_D \\
\text{s.t.} \quad & y_u + y_v + \sum_{\substack{D \in \mathcal{C}: \\ e \in D}} Y_D \le c_e && \text{for all } e = \{u, v\} \in E \\
& Y_D \ge 0 && \text{for all } D \in \mathcal{C}
\end{aligned}$$



Geometric Interpretation of the Dual LP (cont.)

- Consider complete graph with even number of nodes corresponding to points in plane and edge weights given by Euclidean distances.
- By Theorems 13.24 and **??**, there is a perfect matching and a nested dual solution $(y, Y) \geq 0$ satisfying complementary slackness.
- For all nodes $v \in V$, construct (non-overlapping) circular disks D_v centered at v, having radius $y_v \geq 0$. We call D_v the control zone of v.

Definition 13.30 (Moat).

A moat $N \setminus \text{interior}(I)$ is given by a pair of compact sets (N, I) with $I \subseteq N$, $|I \cap V|$ odd, and $V \cap \big(N \setminus \text{interior}(I)\big) = \emptyset$. The width of a moat is

$$\min_{x \in I, y \notin \text{interior}(N)} \|x - y\|_2 \enspace .$$

- Since (y, Y) is feasible, there is room between each pair of nodes $\{u, v\}$ to pack control zones of radius y_u and y_v plus moats of width $Y_{\delta(S)}$ surrounding each odd set $S \subseteq V$ separating u and v.

How to Construct Moats

- Consider odd subsets $S \subset V$ in order of increasing size.
- For each $v \in S$, construct a disk B_v centered at v of radius

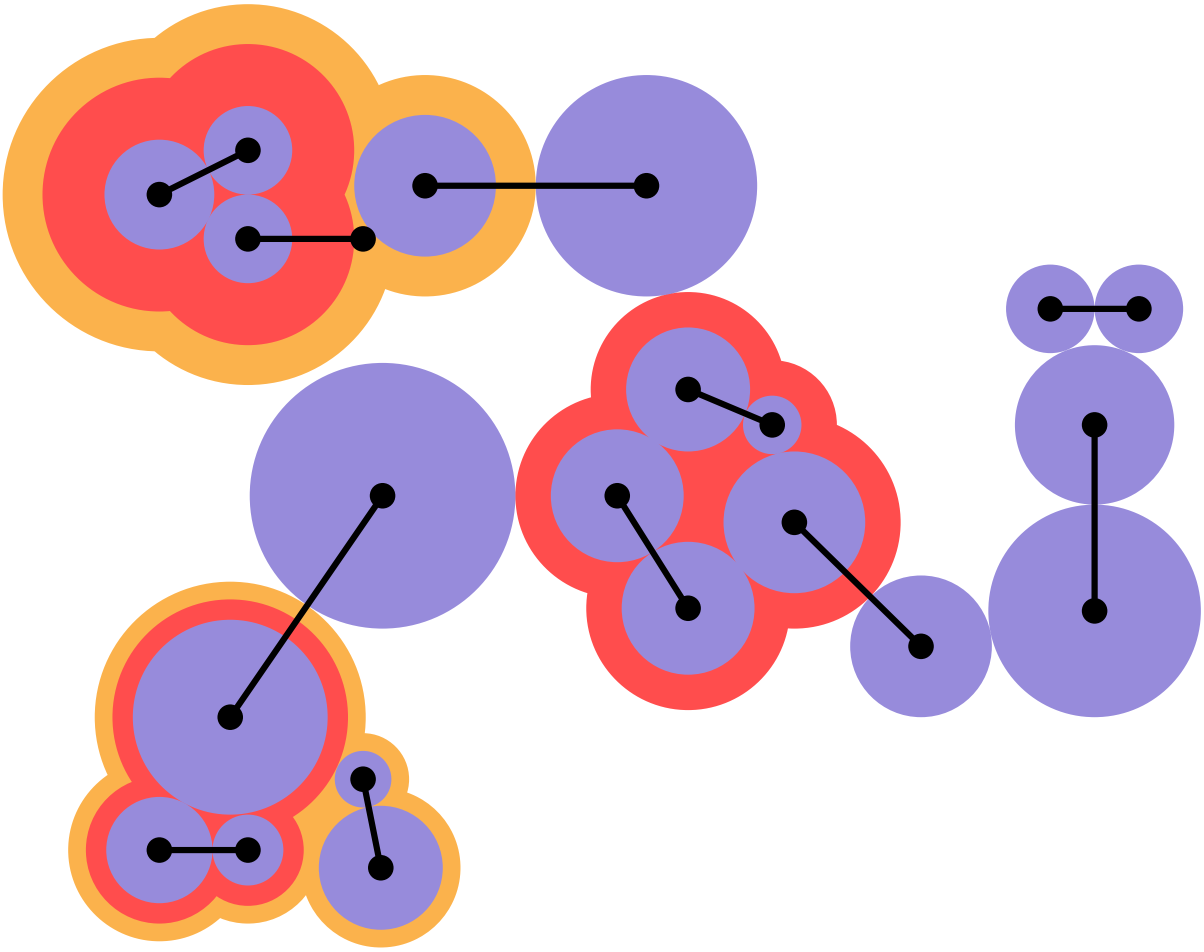
$$\rho_v := y_v + \sum_{v \in T \subsetneq S} Y_{\delta(T)}$$

and a disk B'_v of radius $\rho_v + Y_{\delta(S)}$.
- The moat corresponding to S is given by

$$N := \bigcup_{v \in S} B'_v \qquad \text{and} \qquad I := \bigcup_{v \in S} B_v \enspace .$$

- By the properties of (y, Y) and by construction, the moats and control zones do not overlap.

Example



Goemans-Williamson Algorithm for Minimum Weight Perfect Matchings

Given: Complete graph $G = (V, E)$ with $|V|$ even and edge weights $c \in \mathbb{R}^E_{\geq 0}$ that satisfy the triangle inequality.

Notation:

- An even spanning forest of G is a spanning forest where all trees have an even number of nodes.
- An edge e of an even spanning forest is called even if removing e results in another even forest. Otherwise, e is called odd.

The algorithm of Goemans and Williamson computes an even spanning forest instead of a perfect matching.

Lemma 13.31.

Given an even spanning forest of G, one can construct a perfect matching whose weight is no greater than the forest's weight.

Proof:. . . □

Goemans-Williamson Algorithm (cont.)

Idea: At each stage, the algorithm has a forest F (not even until the end) and a dual feasible solution $(y, Y) \geq 0$.

Notation: Let $\mathcal{T}$ denote the node sets of the trees in F. For each set $T \in \mathcal{T}$, let

$$\text{parity}(T) := \begin{cases} 0 & \text{if } |T| \text{ is even,} \\ 1 & \text{if } |T| \text{ is odd.} \end{cases}$$

Goemans-Williamson Algorithm:

1. Let $F := (V, \emptyset)$, $\mathcal{T} := \{\{v\} \mid v \in V\}$, $y := 0$, and $Y := 0$;
2. If $|T|$ is even for all $T \in \mathcal{T}$, remove any even edges from F and stop;
3. Find an edge $e = \{v, w\}$ with $v \in T_i \in \mathcal{T}$, $w \in T_j \in \mathcal{T}$, $T_i \neq T_j$, that minimizes $\varepsilon := \bar{c}_e / \big(\text{parity}(T_i) + \text{parity}(T_j)\big)$;
4. For each $T \in \mathcal{T}$ such that $T = \{v\}$ for some $v \in V$, add ε to y_v;
5. For each $T \in \mathcal{T}$ with $|T| > 1$ odd, add ε to $Y_{\delta(T)}$;
6. Add e to F and update $\mathcal{T}$ by removing T_i and T_j and adding $T_i \cup T_j$;
7. Go to 2;

Analysis of Goemans-Williamson Algorithm

Theorem 13.32.

The algorithm terminates with an even forest F^* and a feasible dual solution (y^*, Y^*) such that the weight of F^* is at most twice the value of the dual solution (y^*, Y^*).

Proof:... □

Remarks:

- Notice that the Goemans-Williamson Algorithm is considerably more efficient than the Blossom Algorithm.
- It is a primal-dual approximation algorithm with performance guarantee 2.
- The algorithm is a nice example of a more general class of primal-dual approximation algorithms for various problems in combinatorial optimization.