Strengthened LP and its Dual

Primal and dual LP:

$$\begin{array}{llll} \min & \sum_{e \in E} c_e \cdot x_e & & \\ \text{s.t.} & x(\delta(v)) = 1 & \forall v \in V \\ & x(D) \geq 1 & \forall D \in \mathcal{C} \\ & x \geq 0 & \end{array}$$

$$\begin{array}{lll} \max & \sum_{v \in V} y_v + \sum_{D \in \mathcal{C}} Y_D & \\ \text{s.t.} & y_u + y_v + \sum_{D \in \mathcal{C}: e \in D} Y_D \leq c_e & \forall e = \{u, v\} \in E \\ & Y_D \geq 0 & \forall D \in \mathcal{C} \end{array}$$

Notation and remarks:

- For $y \in \mathbb{R}^V$, $Y \in \mathbb{R}^{\mathcal{C}}$, and $e = \{u, v\} \in E$ let

$$\bar{c}_e := c_e - \Big(y_u + y_v + \sum_{D \in \mathcal{C}: e \in D} Y_D \Big) \qquad \text{(reduced cost of edge } e\text{)}.$$

- (y, Y) is a feasible dual solution if and only if $\bar{c}_e \geq 0$ for all $e \in E$ and $Y_D \geq 0$ for all $D \in \mathcal{C}$.

Matching Polytope Theorem (Edmonds 1965)

Theorem 13.19 (Matching Polytope Theorem).

A graph G has a perfect matching if and only if the strengthened primal LP is feasible. Moreover, the minimum weight of a perfect matching is equal to the optimum LP value.

- The proof follows from the Blossom Algorithm for finding minimum-weight perfect matchings that we are going to present next.
- It is again a primal-dual algorithm constructing a perfect matching and a dual solution fulfilling complementary slackness conditions.

Complementary slackness conditions:

$$\begin{array}{lll} x_e > 0 \implies \bar{c}_e = 0 & \text{for all } e \in E, \\ Y_D > 0 \implies x(D) = 1 & \text{for all } D \in \mathcal{C}. \end{array}$$

If x is characteristic vector of perfect matching M, this is equivalent to

$$M \subseteq E_{=} := \{e \in E \mid \bar{c}_e = 0\},$$

$$Y_D > 0 \implies |M \cap D| = 1 \qquad \text{for all } D \in \mathcal{C}.$$

How to Work with Dual Variables Y_D?

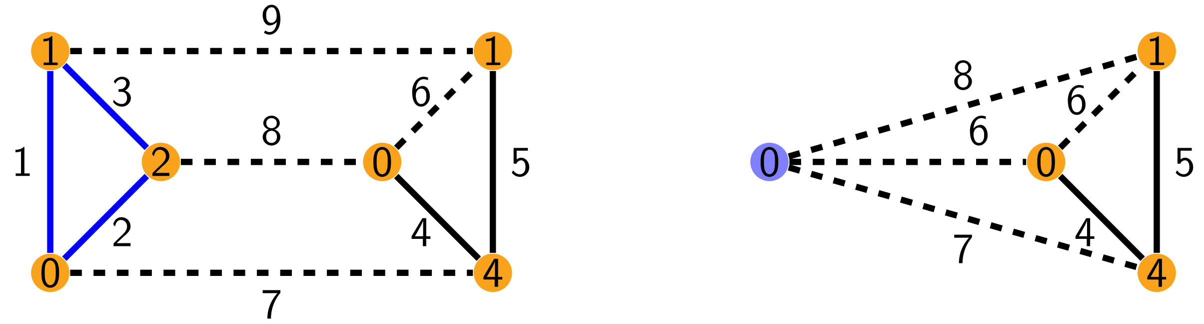
Answer is suggested by Edmond's Matching Algorithm:

- Will be working with derived graphs G' of G that result from (repeated) blossom shrinkings.
- Important observation: Every odd cut of G' is an odd cut of G.
- In particular, if v is a pseudo-node in G' (resulting from (repated) blossom shrinkings), then $\delta_{G'}(v)$ is an odd cut of G.
- These will be the only odd cuts D of G' with positive value Y_D.
- Thus, handle Y_D by replacing it with y_v in G' and requiring $y_v \geq 0$.

Rough Idea of Algorithm

- Find perfect matching $M \subseteq E_=$ with $|M \cap D| = 1$ for all $D \in \mathcal{C}$ with $Y_D > 0$ using tree-extension and augmentation. (cp. bipartite case!)
- But: when changing y, take care of edges joining two outer nodes. If such an edge is in $E_=$, then shrink the corresponding M-blossom C.
- Problem: how to take into account variables y_v, $v \in V(C)$, when shrinking C?
- Solution: replace c_e by $c'_e := c_e - y_v$ for each $e = \{v, w\} \in E$ with $v \in V(C)$, $w \notin V(C)$ and set $y_C := 0$. Notice that reduced costs $\bar{c}_e$ remain unchanged.

Example of shrinking and updating weights:



Obtaining a Solution for (G, c) from a Solution to (G', c')

- Let (G', c') be obtained from (G, c) by shrinking the odd circuit C of equality edges w.r.t. dual-feasible y.
- Let M' be a perfect matching of G' and (y', Y') dual-feasible solution such that M', (y', Y') satisfy complementary slackness and $y'_C \geq 0$.
- Let M be the perfect matching of G extending M' with edges from C.
- Define (y, Y) as follows:
  - For $v \in V(C)$, take y_v as defined before shrinking C.
  - For $v \in V \setminus V(C)$, let $y_v := y'_v$.
  - Let $Y_{\delta(C)} := y'_C$.
  - For $D \in \mathcal{C}$: if $Y'_D > 0$, then $Y_D := Y'_D$; else $Y_D := 0$.

Lemma 13.20.
(y, Y) as defined above is a dual-feasible solution which, together with M, satisfies complementary slackness conditions. □

Carefully Changing y

Let (G', c') be a derived graph with dual-feasible y'; let $M' \subseteq E'_=$ be a matching of G' and T an M'-alternating tree consisting of edges in $E'_=$.

$$\varepsilon_1 := \min\{\bar{c}'_e \mid e \text{ joins an outer node to a node} \notin V(T)\}$$

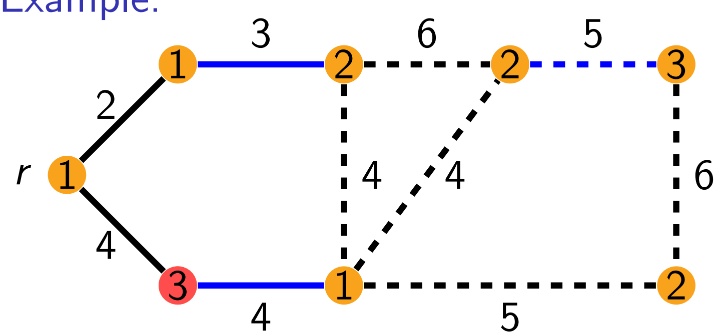
$$\varepsilon_2 := \min\{\bar{c}'_e/2 \mid e \text{ joins two outer nodes}\}$$

$$\varepsilon_3 := \min\{y_v \mid v \text{ is an inner pseudo-node}\}$$

$$\varepsilon := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$$

- for all outer nodes v let $y_v := y_v + \varepsilon$.
- for all inner nodes w let $y_w := y_w - \varepsilon$.

Example:

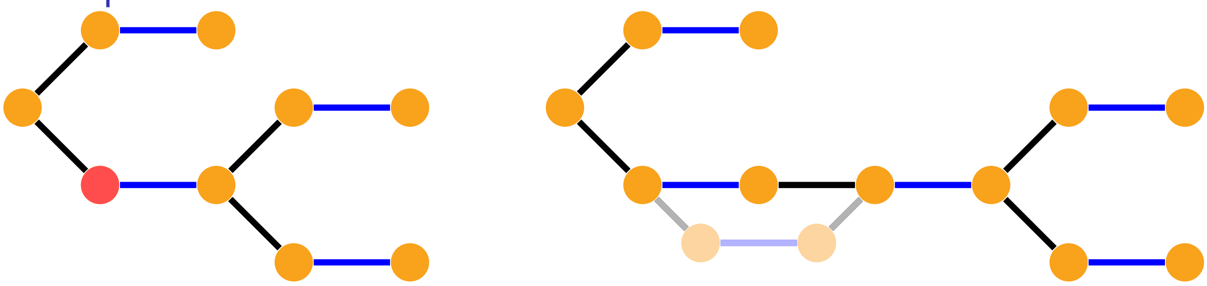


$$\varepsilon_1 := \min\{2, 1, 2\} = 1$$
$$\varepsilon_2 := \min\{\tfrac{1}{2}\} = \tfrac{1}{2}$$
$$\varepsilon_3 := \min\{3\} = 3$$
$$\varepsilon := \min\{1, \tfrac{1}{2}, 3\} = \tfrac{1}{2}$$

Expansion of Pseudo-nodes

- After finding an augmentation, we cannot expand all pseudo-nodes as in Edmond's Algorithm.
- For pseudo-node v with $y_v > 0$, the expansion yields variable $Y_D > 0$ for cut $D \in \mathcal{C}$ that is not of form $\delta_{G'}(u)$ for some pseudo-node u.
- Thus, pseudo-node v can only be expanded if $y_v = 0$.
- In this case, we *need* to do it as a dual variable change might not enable us to augment, extent, or shrink (if $\varepsilon = \varepsilon_3$).
- Thus, we need to expand carefully in order not to lose the progress that has been made, i. e., we also need to update T.

Example:



Blossom Algorithm for Minimum-Weight Perfect Matching

Let y be a dual-feasible solution, $M' \subseteq E_=$ matching, $G' := G$;
let $T := (\{r\}, \emptyset)$, where r is an M'-exposed node of G';

1 if there is an outer node $v \in V(T)$ and $w \notin V(T)$ with $\{v, w\} \in E'_=$
2   if w is M'-exposed
3   use edge $\{v, w\}$ to augment M';
4   if there is no M'-exposed node in G'
5   extend M' to perfect matching M of G and stop; (M optimal)
6   else replace T by $(\{r\}, \emptyset)$ where r is M'-exposed; go to 1;
7   else use $\{v, w\}$ to extend tree T; go to 1;
8 if there are outer nodes $v, w \in V(T)$ with $\{v, w\} \in E_=$
9   use $\{v, w\}$ to shrink and update M', T, and c'; go to 1;
10 if there is an inner pseudo-node $v \in V(T)$ with $y_v = 0$
11   expand v and update M', T, and c'; go to 1;
12 if for all outer nodes v and $\{v, w\} \in E'$ node w is inner
13   if there is no inner pseudo-node, then stop; (no perfect matching)
14 change y and go to 1;

Correctness and Running Time

Theorem 13.21.

The Blossom Algorithm terminates after $O(n)$ augmentation steps, and $O(n^2)$ tree-extension, shrinking, expanding and dual change steps. Moreover, it returns a minimum-weight perfect matching, or determines correctly that G has no perfect matching.

Proof: See book of Cook et al., proof of Theorem 5.16. □

Theorem 13.22.

The Blossom Algorithm can be implemented to run in $O(n^2 m)$ time. □

Optimal Dual Solutions of Matching Problems

Definition 13.23 (Nested sets).

Let N be a set and $\mathcal{F} \subseteq 2^N$ a family of subsets of N. Then $\mathcal{F}$ is called *nested* if, for all $A, B \in \mathcal{F}$ it holds that $A \subseteq B$ or $B \subseteq A$ or $A \cap B = \emptyset$.

Theorem 13.24.

If there exists a minimum-weight perfect matching, then the dual linear program has an optimal solution (y, Y) that is nested, i. e., the family of subsets $\mathcal{S} := \{S \subseteq V \mid Y_{\delta(S)} > 0\}$ is nested.

Proof: Consider the solution found by the Blossom Algorithm.

- Notice that each $S \in \mathcal{S}$ corresponds to a pseudo-node v of a derived graph G' of G.
- Moreover, the final derived graph G'' is a derived graph of G'. □

Optimal Dual Solutions of Matching Problems (cont.)

Theorem 13.25.

Let $G = (V, E)$ a graph and $c \in \mathbb{Z}^E$. If there is a minimum-weight perfect matching for (G, c) and weight $c(E(C))$ of every circuit C of G is even, then there is an optimal dual solution (y, Y) that is nested and integral.

Proof: See book of Cook et al., proof of Theorem 5.18. □

Corollary 13.26.

Let $G = (V, E)$ a graph and $c \in \mathbb{Z}^E$. If there is a minimum-weight perfect matching for (G, c), then there is an optimal dual solution (y, Y) that is nested and half-integral. □