Minimum-Weight Perfect Matchings

Given: graph $G = (V, E)$, edge weights c_e, $e \in E$;

Task: find perfect matching M in G of minimum weight $c(M) = \sum_{e \in M} c_e$.

IP formulation:

$$\begin{array}{lll} \min & \sum_{e \in E} c_e \cdot x_e & \\ \text{s.t.} & x\big(\delta(v)\big) = 1 & \text{for all } v \in V \\ & x_e \in \{0,1\} & \text{for all } e \in E \end{array}$$

The LP relaxation is obtained by replacing $x_e \in \{0,1\}$ with $x_e \geq 0$.

Remarks:

- The complete graph on three nodes K_3 shows that the LP relaxation can be feasible even though there is no perfect matching.
- It is easy to construct instances for which the optimum LP value is strictly smaller than the minimum weight of a perfect matching.

Birkhoff's Theorem

Theorem 13.17 (Birkhoff's Theorem).

Let G be bipartite. Then G has a perfect matching if and only if the LP relaxation is feasible. Moreover, the minimum weight of a perfect matching is equal to the optimum LP value.

Remarks:

- Birkhoff's Theorem can be easily proved via network flow arguments.
- It also follows from an algorithm for finding minimum weight perfect matchings in bipartite graphs that we are going to present next.
- It is a primal-dual algorithm which constructs a perfect matching and a dual solution fulfilling the complementary slackness conditions.

Dual Linear Program and Complementary Slackness

Primal and dual LP:

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e \cdot x_e \\ \text{s.t.} \quad & x\big(\delta(v)\big) = 1 \quad \forall v \in V \\ & x \geq 0 \end{aligned} \qquad \begin{aligned} \max \quad & \sum_{v \in V} y_v \\ \text{s.t.} \quad & y_u + y_v \leq c_e \quad \forall e = \{u, v\} \in E \end{aligned}$$

Notation and remarks:

- For $y \in \mathbb{R}^V$ and $e = \{u, v\} \in E$ let $\bar{c}_e := c_e - (y_u + y_v)$.
- $y \in \mathbb{R}^V$ is a feasible dual solution if and only if $\bar{c}_e \geq 0$ for all $e \in E$.
- In this case, let $E_= := \{e \in E \mid \bar{c}_e = 0\}$.

Complementary slackness conditions:

$$x_e > 0 \quad \Longrightarrow \quad \bar{c}_e = 0 \qquad \text{for all } e \in E.$$

If x is characteristic vector of perfect matching M, this is equivalent to

$$M \subseteq E_= \ .$$

Hungarian Algorithm (Kuhn 1955; Munkres 1957)

Let $G = (V, E)$ bipartite, y feasible dual solution and matching $M \subseteq E_=$.

1 stop if M is perfect; else let r an M-exposed node and $T := (\{r\}, \emptyset)$;

2 while there is outer node $v \in V(T)$ and $w \notin V(T)$ with $\{v, w\} \in E_=$;

3 if w is M-exposed, then use $\{v, w\}$ to augment M; go to step 1.;

4 else use $\{v, w\}$ to extend tree T;

5 if for all outer nodes v and $\{v, w\} \in E$ node w is inner, then stop;
(G has no perfect matching as the inner nodes form a Tutte set)

6 let $\varepsilon := \min\{\bar{c}_{\{v,w\}} \mid v \text{ outer}, w \notin V(T)\}$;
for all outer nodes v let $y_v := y_v + \varepsilon$;
for all inner nodes w let $y_w := y_w - \varepsilon$;
go to step 2.;

Remark: To start, one can, e. g., set $y := \frac{1}{2} \min_{e \in E} c_e$ and $M := \emptyset$.

Correctness and Running Time

- y is a feasible dual solution throughout the algorithm.
- $M \subseteq E_=$ and $E(T) \subseteq E_=$ throughout the algorithm.
- Thus, if the algorithm terminates with perfect matching, it is optimal.
- In every iteration, one of the following actions is taken:
  - matching M is augmented such that $|M|$ increases;
  - tree T is extended;
  - an edge joining an outer node v to a node $w \notin V(T)$ enters $E_=$, leading to an augmentation or tree-extension in next iteration.

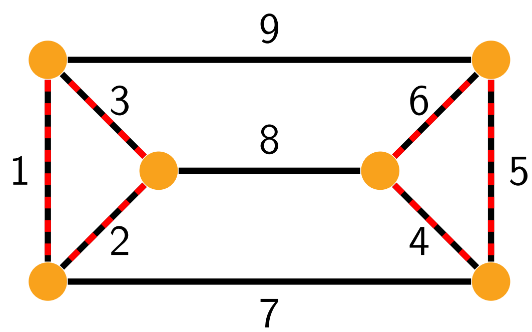
Theorem 13.18.
The Hungarian Algorithm solves the minimum-weight perfect matching problem in $O(n^2 m)$ time. □

Remark: The running time can be improved to $O(n^3)$ or to $O\big(nS(n,m)\big)$ time where $S(n,m)$ is the running time of a shortest-path algorithm.

Strengthening the LP for General Graphs

The result of Birkhoff's Theorem fails in general:

Example:



- The minimum weight of a perfect matching is 14.
- There is a feasible LP solution of value 10.5
- If we strengthen the LP by adding the requirement that the variables of the three horizontal edges sum up to at least 1, the optimal LP value is 14.

Idea: Strengthen the LP relaxation by adding blossom inequalities:

$$x\big(\delta(S)\big) \geq 1 \qquad \text{for all } S \subseteq V \text{ with } |S| \text{ odd.}$$

Notation: Let

$$\mathcal{C} := \big\{\delta(S) \mid S \subseteq V,\ |S| \text{ odd}\big\} \setminus \big\{\delta(v) \mid v \in V\big\} \ .$$