Blossom Shrinking Lemma

Definition 13.12 (Blossom).

Let M be a matching in G. An M-blossom in G is a factor-critical subgraph C of G with $|M \cap E(C)| = \frac{1}{2}(|V(C)| - 1)$. The (unique) M-exposed node in C is the base of C.

Let C be an M-blossom in G. Suppose there is an M-alternating v-r-path Q of even length from an M-exposed node v to the base r of C, where $E(Q) \cap E(C) = \emptyset$.

Lemma 13.13 (Blossom Shrinking Lemma).

Let G' and M' result from G and M by shrinking M-blossom C to a single node. Then M is a maximum matching in G if and only if M' is a maximum matching in G'.

Proof:. . . □

Alternating Forests

Definition 13.14.

Given a graph G and a matching M in G, an M-alternating forest is a forest F in G such that

- the roots of trees in F are exactly the M-exposed nodes.
- For any $v \in V(F)$, the unique path $P(v)$ from v to the root of its tree is M-alternating.

A node $v \in V(F)$ with even distance to its root is called outer, all other nodes in $V(F)$ are called inner.

In the following algorithm we will only consider alternating forests where every inner node has degree two. In this case, there is

exactly one more outer node than inner nodes in every tree.

Edmonds' Matching Algorithm

Given: Graph G and matching M in G.
Output: Matching M' with $|M'| = |M| + 1$ or the information that no such matching exists.

1. Let F be the forest consisting of all M-exposed nodes.
2. Let x be an outer node of F and y adjacent to x, but not an inner node of F. Prefer $y \in V(F)$. STOP, if no such nodes x, y exist.
3. If $y \in V(F)$ and y in different tree, augment M on path $P(x) \cup \{x, y\} \cup P(y)$ to obtain the larger matching M'. STOP;
4. If $y \in V(F)$ and y in the same tree, shrink the circuit (M-blossom) in $F \cup \{x, y\}$. Goto 2.
5. If $y \notin V(F)$, then add $\{x, y\}$ and the matching edge covering y to F. Goto 2.

Correctness of Edmonds' Matching Algorithm

Theorem 13.15 (Edmonds).
Edmonds' matching algorithm works correctly.

Proof:. . . □

Theorem 13.16.
A maximum matching can be computed in $\mathcal{O}(|V|^3)$ time.

Remark: Edmonds' Matching Algorithm can be implemented to run in time $O(nm \log n)$.