Chapter 13: Optimal Matchings

(cp. Cook, Cunningham, Pulleyblank, Schrijver, Chapter 5; or Korte, Vygen, Chapter 10)

Maximum Matching and Minimum Node Cover

Definition 13.1.

Consider an undirected graph $G = (V, E)$ and $M \subseteq E$.

i. M is a matching in G if $e \cap e' = \emptyset$ for all $e, e' \in M$ with $e \neq e'$.
ii. A node $v \in V$ is covered by M if $v \in e$ for some $e \in M$.
iii. $v \in V$ is M-exposed if $v \notin e$ for all $e \in M$, i. e., v *not* covered by M.
iv. A matching M is perfect if each node is covered by M.

The maximum matching problem asks for a matching of maximal cardinality in G.

Recall: In a bipartite graph $G = (A \dot\cup B, E)$, a maximum cardinality matching can be found by a maximum flow computation.

Recall: $C \subseteq V$ is a node cover if $e \cap C \neq \emptyset$ for all $e \in E$.

Observation: Let $\nu(G)$ denote the size of a maximum matching, and $\tau(G)$ denote the size of a minimum node cover in G. Then $\nu(G) \leq \tau(G)$.

Theorems of Kőnig and Hall

Recall: If G is a bipartite graph, then $\nu(G) = \tau(G)$ (Kőnig's Theorem).

Definition 13.2.

In general, G is called Kőnig-Egerváry graph if $\nu(G) = \tau(G)$.

For $X \subseteq V$, the neighborhood $\Gamma(X)$ of X is defined as

$$\Gamma(X) := \{v \in V \setminus X \mid \{v, w\} \in E \text{ for some } w \in X\} \ .$$

Theorem 13.3 (Hall).

Let G be a bipartite graph with bipartition $V = A \dot\cup B$. Then G has a matching covering A if and only if $|\Gamma(X)| \geq |X|$ for all $X \subseteq V$.

Proof:. . . □

Corollary 13.4.

Let G be a bipartite graph with bipartition $V = A \dot\cup B$. Then G has a perfect matching if and only if $|A| = |B|$ and $|\Gamma(X)| \geq |X|$ for all $X \subseteq V$.

Berge's Augmenting Path Theorem

Recall: A flow x is maximum if and only if there is no x-augmenting path.

Definition 13.5.

Let M be a matching in G (not necessarily bipartite).

i. Path P is M-alternating if $E(P) \setminus M$ is a matching.
ii. M-alternating path is M-augmenting if its endpoints are M-exposed.

Note: Let M be a matching of some bipartite graph G and let x be the integral feasible flow corresponding to the matching M. There is a 1-1 correspondence between x-augmenting and M-alternating paths.

Theorem 13.6 (Berge).

A matching M in G (not necessarily bipartite) is maximum if and only if there is no M-augmenting path.

Proof:. . . □

Tutte's Theorem

Definition 13.7.

For $X \subseteq V(G)$ let $q_G(X) := \#\{\text{odd connected components in } G - X\}$.

i. G satisfies the Tutte condition if $q_G(X) \leq |X|$ for all $X \subseteq V$.
ii. $X \neq \emptyset$ is a barrier if $q_G(X) = |X|$.

Theorem 13.8 (Tutte).

Graph G has perfect matching if and only if it satisfies the Tutte condition.

Exercise: Show that $q_G(X) - |X| \equiv |V| \bmod 2$ for all $X \subseteq V$.

Definition 13.9.

Graph G is factor-critical if $G - v$ has a perfect matching for all $v \in V(G)$.
A matching M is near-perfect if it covers all nodes but one.

Proof (of Tutte's Theorem):... ☐

Certificates for Maximum Matchings

Tutte's Theorem gives a good characterization of the perfect matching problem: either G has a perfect matching, or there is $X \subseteq V$ with $q_G(X) > |X|$ (Tutte set) proving that no perfect matching in G exists.

Observation 13.10.

For $X \subseteq V$, any matching leaves at least $q_G(X) - |X|$ vertices uncovered.

Theorem 13.11 (Berge).

$$\nu(G) = \tfrac{1}{2}\left(|V| - \max_{X \subseteq V}(q_G(X) - |X|)\right) .$$

Proof:... ☐

Generic Matching Algorithm

1. Start with empty matching;
2. Find augmenting path P and augment;
3. Goto 2.;

Question: How to find an augmenting path? (Easy in bipartite graphs!)