Potential Reduction Algorithm: Potential Function

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ and consider primal-dual pair of LPs:

$$\begin{array}{rl} \min & c^T \cdot x \\ \text{s.t.} & A \cdot x = b \\ & x \geq 0 \end{array} \qquad\qquad \begin{array}{rl} \max & p^T \cdot b \\ \text{s.t.} & p^T \cdot A + s^T = c^T \\ & s \geq 0 \end{array}$$

Assume that rows of A are linearly independent and there exists $x^0 > 0$ and (p^0, s^0) with $s^0 > 0$ which are feasible for primal and dual LP, respectively.

Idea: Stay away from the boundary!

Potential function:

$$G(x,s) := q \log(s^T \cdot x) - \sum_{j=1}^{n} \log x_j - \sum_{j=1}^{n} \log s_j$$

where q is a constant larger than n.

Potential Reduction Algorithm: Duality Gap

If x and (p,s) are feasible, then the duality gap is

$$c^T \cdot x - b^T \cdot p = (s^T + p^T \cdot A) \cdot x - x^T \cdot A^T \cdot p = s^T \cdot x$$

Theorem 12.6.

An algorithm that maintains primal and dual feasibility and reduces $G(x,s)$ by at least $\delta > 0$ at each iteration, finds solutions with duality gap $(s^k)^T \cdot x^k \leq \varepsilon$ after

$$K := \left\lceil \frac{G(x^0, s^0) + (q-n)\log(1/\varepsilon) - n \log n}{\delta} \right\rceil$$

iterations.

Proof: . . . □

Potential Reduction Algorithm: Basic Idea

- Start with feasible $x > 0$ and (p, s) with $s > 0$.
- Try to find direction d such that

$$G(x+d, s) < G(x, s) \quad \text{and} \quad A \cdot d = 0, \quad \|X^{-1} \cdot d\|_2 \leq \beta < 1 \ .$$

- Minimizing $G(x+d, s)$ s.t. $A \cdot d = 0$, $\|X^{-1} \cdot d\|_2 \leq \beta$ is difficult due to objective function (non-linear, non-convex).
- Linearize the objective function by taking the first order Taylor series expansion in d:

$$\begin{aligned} \min \quad & \left(\nabla_x G(x,s)\right)^T \cdot d \\ \text{s.t.} \quad & A \cdot d = 0 \\ & \|X^{-1} \cdot d\| \leq \beta \end{aligned}$$

Note: Same as in affine scaling except for different objective funct. $\hat{c}$

$$\hat{c}_i := \frac{\partial G(x,s)}{\partial x_i} = \frac{q \cdot s_i}{s^T \cdot x} - \frac{1}{x_i} \ .$$

Potential Reduction Algorithm: Basic Idea (cont.)

- Applying Lemma 12.3 with $Y := X$ and $c := \hat{c}$, we obtain optimal direction

$$d^* := -\beta \cdot X \cdot \frac{u}{\|u\|_2}$$

with $u := X \cdot \left(\hat{c} - A^T \cdot (A \cdot X^2 \cdot A^T)^{-1} \cdot A \cdot X^2 \cdot \hat{c}\right)$.

- Since $X \cdot \hat{c} = \dfrac{q}{s^T \cdot x} \cdot X \cdot s - e$, we obtain

$$u = \left(I - X \cdot A^T \cdot (A \cdot X^2 \cdot A^T)^{-1} \cdot A \cdot X\right) \cdot \left(\frac{q}{s^T \cdot x} \cdot X \cdot s - e\right) \ .$$

- Moreover, $G(x, s)$ decreases by $\beta \cdot \|u\|_2 + O(\beta^2)$, where the first term comes from Lemma 12.3 and the second is due to omitted higher order terms in the Taylor series expansion of $G(x, s)$.
- Thus, if $\|u\|_2$ is large enough, then $G(x, s)$ decreases by at least δ.
- Otherwise update dual variables.

Potential Reduction Algorithm

1. Let $x^0 > 0$ feasible, (p^0, s^0) with $s^0 > 0$; set $k := 0$; (initialization)
2. If $(s^k)^T \cdot x^k < \varepsilon$, then stop; (optimality test)
3. Let
$$X_k := \text{diag}(x_1^k, \dots, x_n^k)$$
$$\bar{A}^k := (A \cdot X_k)^T \cdot (A \cdot {X_k}^2 \cdot A^T)^{-1} \cdot A \cdot X_k$$
$$u^k := (I - \bar{A}^k) \cdot \left(\frac{q}{(s^k)^T \cdot x^k} \cdot X_k \cdot s^k - e \right)$$
$$d^k := -\beta \cdot X_k \cdot \frac{u^k}{\|u^k\|_2} \qquad \text{(update direction)}$$
4. If $\|u^k\|_2 \geq \gamma$, then $x^{k+1} := x^k + d^k$, $s^{k+1} := s^k$, $p^{k+1} := p^k$; (primal step)
5. If $\|u^k\|_2 < \gamma$, then $x^{k+1} := x^k$, $s^{k+1} := \frac{(s^k)^T \cdot x^k}{q} \cdot {X_k}^{-1} \cdot (u^k + e)$,
$$p^{k+1} := p^k + (A \cdot {X_k}^2 \cdot A^T)^{-1} \cdot A \cdot X_k \cdot \left(X_k \cdot s^k - \frac{(s^k)^T \cdot x^k}{q} \cdot e \right) ;$$
(dual step)
6. Let $k := k + 1$ and go to 2;

Potential Reduction Algorithm: Behavior

For every k, the vectors x^k and (p^k, s^k) are primal and dual feasible solutions, respectively.

Theorem 12.7.

Let $\beta < 1$ and $\gamma < 1$:

a. If $\|u^k\|_2 \geq \gamma$ (primal step), then
$$G(x^{k+1}, s^{k+1}) - G(x^k, s^k) \leq -\beta \cdot \gamma + \frac{\beta^2}{2(1-\beta)} \ .$$

b. If $\|u^k\|_2 < \gamma$ (dual step), then
$$G(x^{k+1}, s^{k+1}) - G(x^k, s^k) \leq -(q - n) + n \log \frac{q}{n} + \frac{\gamma^2}{2(1-\gamma)} \ .$$

c. If $q = n + \sqrt{n}$, $\beta \approx 0.285$ and $\gamma \approx 0.479$, then the potential reduction algorithm reduces $G(x, s)$ by at least $\delta := 0.079$ at each iteration.

Proof: See Bertsimas & Tsitsiklis, proof of Theorem 9.5. □

Potential Reduction Algorithm: Running Time

Initialization: Use auxiliary problem (similar to affine scaling algorithm). For details, see Bertsimas & Tsitsiklis, Sect. 9.3.

Running Time:

- The potential reduction algorithm finds ε-optimal solutions in time

$$O\big(n^{3.5}\log\varepsilon^{-1} + n^5\log(n\,U)\big)$$

  where $U := \max\{|a_{ij}|, |b_i|, |c_j|\}$ (all integer).

- If ε is taken sufficiently small, an optimal solution can be found by rounding. This results in a polynomial (in n and $\log U$) time algorithm.

Further Interior Point Algorithms

- Primal path following algorithm: see Bertsimas & Tsitsiklis, Sect. 9.4.
- Primal-dual path following algorithm: see Bertsimas & Tsitsiklis, Sect. 9.5.