Dantzig-Wolfe Decomposition

Consider a linear program of the form

$$\begin{array}{rrcrcl} \min & {c_1}^T \cdot x_1 & + & {c_2}^T \cdot x_2 & & \\ \text{s.t.} & D_1 \cdot x_1 & + & D_2 \cdot x_2 & = & b_0 \\ & F_1 \cdot x_1 & & & = & b_1 \\ & & & F_2 \cdot x_2 & = & b_2 \\ & & & x_1,\, x_2 & \geq & 0 \end{array}$$

with $c_1 \in \mathbb{R}^{n_1}$, $c_2 \in \mathbb{R}^{n_2}$, $b_0 \in \mathbb{R}^{m_0}$, $b_1 \in \mathbb{R}^{m_1}$, $b_2 \in \mathbb{R}^{m_2}$.

Reformulation of the problem: For $i = 1, 2$, let $P_i := \{x_i \geq 0 \mid F_i \cdot x_i = b_i\}$.

$$\begin{array}{rrcrcl} \min & {c_1}^T \cdot x_1 & + & {c_2}^T \cdot x_2 & & \\ \text{s.t.} & D_1 \cdot x_1 & + & D_2 \cdot x_2 & = & b_0 \\ & x_1 \in P_1 & , & x_2 \in P_2 & & \end{array}$$

- Let x_i^j, $j \in J_i$, be the extreme points of P_i.
- Let w_i^k, $k \in K_i$, be a complete set of extreme rays of P_i.

Dantzig-Wolfe Decomposition: The Master Problem

For $i = 1, 2$, any $x_i \in P_i$ can be written as

$$x_i = \sum_{j \in J_i} \lambda_i^j \cdot x_i^j + \sum_{k \in K_i} \theta_i^k \cdot w_i^k$$

with $\lambda_i^j,\, \theta_i^k \geq 0$ and $\sum_{j \in J_i} \lambda_i^j = 1$.

The reformulation thus leads to the following master problem:

$$\min \sum_{j \in J_1} \lambda_1^j\,(c_1^T x_1^j) + \sum_{k \in K_1} \theta_1^k\,(c_1^T w_1^k) + \sum_{j \in J_2} \lambda_2^j\,(c_2^T x_2^j) + \sum_{k \in K_2} \theta_2^k\,(c_2^T w_2^k)$$

$$\text{s.t.} \sum_{j \in J_1} \lambda_1^j \begin{pmatrix} D_1 x_1^j \\ 1 \\ 0 \end{pmatrix} + \sum_{k \in K_1} \theta_1^k \begin{pmatrix} D_1 w_1^k \\ 0 \\ 0 \end{pmatrix} + \sum_{j \in J_2} \lambda_2^j \begin{pmatrix} D_2 x_2^j \\ 0 \\ 1 \end{pmatrix} + \sum_{k \in K_2} \theta_2^k \begin{pmatrix} D_2 w_2^k \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_0 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_1,\, \lambda_2,\, \theta_1,\, \theta_2 \geq 0$$

The master problem has only $m_0 + 2$ constraints but a huge number of variables. $\longrightarrow$ Employ delayed column generation!

Dantzig-Wolfe Decomposition: Pricing Problem

Let B be a feasible basis to the master problem and $p^T := c_B{}^T \cdot B^{-1}$ the associated dual solution: $p^T = (q^T, r_1, r_2)$ with $q \in \mathbb{R}^{m_0}$, r_1, $r_2 \in \mathbb{R}$.

Compute the reduced cost coefficient of a variable λ_1^j:

$$c_1{}^T \cdot x_1^j - (q^T, r_1, r_2) \cdot \begin{pmatrix} D_1 \cdot x_1^j \\ 1 \\ 0 \end{pmatrix} = (c_1{}^T - q^T \cdot D_1) \cdot x_1^j - r_1$$

Compute the reduced cost coefficient of a variable θ_1^k:

$$c_1{}^T \cdot w_1^k - (q^T, r_1, r_2) \cdot \begin{pmatrix} D_1 \cdot w_1^k \\ 0 \\ 0 \end{pmatrix} = (c_1{}^T - q^T \cdot D_1) \cdot w_1^k$$

In order to solve the pricing problem for variables λ_i^j and θ_i^k, we consider the following LP:

$$\min \ (c_i{}^T - q^T \cdot D_i) \cdot x_i \quad \text{s.t.} \ \ x_i \in P_i$$

This is called the *i*th subproblem.

Dantzig-Wolfe Decomposition: Pricing Problem (cont.)

Consider *i*th subproblem: $\min \ (c_i{}^T - q^T \cdot D_i) \cdot x_i \quad \text{s.t.} \ \ x_i \in P_i$

Case 1: *i*th subproblem is unbounded:
$\Longrightarrow$ simplex algorithm yields extreme ray w_i^k with $(c_i{}^T - q^T \cdot D_i) \cdot w_i^k < 0$
$\Longrightarrow$ reduced cost of θ_i^k is negative
$\longrightarrow$ generate column $\begin{pmatrix} D_i w_i^k \\ 0 \\ 0 \end{pmatrix}$ and let it enter the basis in master problem.

Case 2: *i*th subproblem has finite optimal cost $< r_i$:
$\Longrightarrow$ simplex algorithm yields extreme point x_i^j with $(c_i{}^T - q^T \cdot D_i) \cdot x_i^j < r_i$
$\Longrightarrow$ reduced cost of λ_i^j is negative
$\longrightarrow$ generate column $\begin{pmatrix} D_i x_i^j \\ \vdots \end{pmatrix}$ and let it enter the basis in master problem.

Case 3: *i*th subproblem has finite optimal cost $\geq r_i$:

$$\Longrightarrow \quad \begin{array}{ll} (c_i{}^T - q^T \cdot D_i) \cdot x_i^j \geq r_i & \text{for all } j \in J_i \text{ and} \\ (c_i{}^T - q^T \cdot D_i) \cdot w_i^k \geq 0 & \text{for all } k \in K_i. \end{array}$$

$\Longrightarrow$ Variables λ_i^j and θ_i^k have reduced cost ≥ 0.

Dantzig-Wolfe Decomposition: Summary

- The given problem is transformed into an equivalent problem with few constraints but many variables.
- The pricing problem can be solved by solving smaller LPs over the polyhedra P_i.

Economic interpretation: Organization with two divisions and common objective $D_1 \cdot x_1 + D_2 \cdot x_2 = b_0$.

- Central planner assigns values q for each unit of contribution towards common objective.
- Division i wants to minimize ${c_i}^T \cdot x_i$ s.t. its own constraint $x_i \in P_i$.
- Since x_i contributes $D_i \cdot x_i$ towards common objective, the overall objective for division i is $\min({c_i}^T - q^T \cdot D_i) \cdot x$.
- The divisions propose solutions to the central planner who combines them with previous solutions and comes up with new values q.

Dantzig-Wolfe Decomposition: Generalization

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{t} {c_i}^T \cdot x_i \\
\text{s.t.} \quad & \sum_{i=1}^{t} D_i \cdot x_i = b_0 \\
& F_i \cdot x_i = b_i \qquad \text{for } i = 1, \dots, t \\
& x_1, \dots, x_t \geq 0
\end{aligned}
$$

- Proceed as before $\longrightarrow$ t subproblems for pricing.
- Sometimes even useful for $t = 1$.

Dantzig-Wolfe Decomposition: Phase I

How to find an initial basic feasible solution?

- Use phase I of simplex method to find an extreme point x_i^1 of P_i, for $i = 1, \dots, t$.
- W.l.o.g. $\sum_{i=1}^{t} D_i \cdot x_i^1 \leq b_0$. Introduce slack variables $y \in \mathbb{R}^{m_0}$ and solve auxiliary master problem:

$$
\begin{aligned}
\min \quad & \sum_{s=1}^{m_0} y_s \\
\text{s.t.} \quad & \sum_{i=1}^{t} \Big(\sum_{j \in J_i} \lambda_i^j (D_i \cdot x_i^j) + \sum_{k \in K_i} \theta_i^k (D_i \cdot w_i^k) \Big) + y = b_0 \\
& \sum_{j \in J_i} \lambda_i^j = 1 \qquad \text{for } i = 1, \dots, t \\
& \lambda,\ \theta,\ y \geq 0
\end{aligned}
$$

Dantzig-Wolfe Decomposition: Example

Arc-based LP formulation of min-cost multi-commodity flow problem:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{t} \Big(\sum_{a \in A} c(a) \cdot x_{i,a} \Big) \\
\text{s.t.} \quad & \sum_{i=1}^{t} x_{i,a} \leq u(a) \qquad \text{for } a \in A \\
& \sum_{a \in \delta^-(v)} x_{i,a} - \sum_{a \in \delta^+(v)} x_{i,a} = \begin{cases} d_i & \text{if } v = t_i \\ -d_i & \text{if } v = s_i \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } i = 1, \dots, t \\
& x \geq 0
\end{aligned}
$$

- For $i = 1, \dots, t$, let $P_i := \{x_i \mid x_i \text{ is } s_i\text{-}t_i\text{-flow of value } d_i\}$.
- Extreme points of polyhedron P_i: s_i-t_i-path flows of value d_i (denoted by x_i^P for s_i-t_i-path $P \in \mathcal{P}_i$)

Dantzig-Wolfe Decomposition: Example (cont.)

Master problem:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{t} \Big(\sum_{P \in \mathcal{P}_i} \lambda_i^P \cdot (d_i \, c_P) \Big) & \\
\text{s.t.} \quad & \sum_{i=1}^{t} \Big(\sum_{P \in \mathcal{P}_i : a \in P} \lambda_i^P \cdot d_i \Big) \leq u(a) & \text{for } a \in A \\
& \sum_{P \in \mathcal{P}_i} \lambda_i^P = 1 & \text{for } i = 1, \dots, t \\
& \lambda \geq 0 &
\end{aligned}
$$

- Setting $x_P := \lambda_i^P \cdot d_i$ for $P \in \mathcal{P}_i$ yields the path-based LP formulation!
- The ith subproblem (pricing problem for variables λ_i^P, $P \in \mathcal{P}_i$) is a shortest s_i-t_i-path problem.