Local Sensitivity Analysis: Adding New Equation

Add a new equality ${a_{m+1}}^T \cdot x = b_{m+1}$ to the primal LP.

Observations:

- If x^* satisfies the new constraint, it remains optimal.
- Otherwise, w.l.o.g. ${a_{m+1}}^T \cdot x > b_{m+1}$; consider auxiliary problem (with huge M):

$$\begin{aligned} \min \;\; & c^T \cdot x + M \cdot x_{n+1} \\ \text{s.t.} \;\; & A \cdot x = b \\ & {a_{m+1}}^T \cdot x - x_{n+1} = b_{m+1} \\ & (x, x_{n+1}) \geq 0 \end{aligned}$$

- New feasible basis $\bar{B} := \begin{pmatrix} B & 0 \\ {a_{\dots}}^T & -1 \end{pmatrix}$ with $\bar{B}^{-1} = \begin{pmatrix} B^{-1} & 0 \\ {a_{\dots}}^T \cdot B^{-1} & -1 \end{pmatrix}$ and associated basic feasible solution $(x^*, {a_{m+1}}^T \cdot x^* - b_{m+1})$.
- Apply the primal simplex algorithm to reoptimize!

Local Sensitivity Analysis: Changing Right-Hand Side

Change b to $b + \delta \cdot e_i$, that is, only b_i is changed to $b_i + \delta$.

Observations:

- Optimality condition $\bar{c} \geq 0$ is not affected!
- Feasibility: $B^{-1} \cdot (b + \delta \cdot e_i) \geq 0$?
- Let $g := (\beta_{1i}, \dots, \beta_{mi})^T$ be the ith column of B^{-1}:

$$B^{-1} \cdot (b + \delta \cdot e_i) = x_B^* + \delta \cdot g \geq 0$$

$$\iff \quad x_{B(j)}^* + \delta \cdot \beta_{ji} \geq 0 \qquad \text{for } j = 1, \dots, m$$

$$\iff \quad \max_{j:\beta_{ji}>0} -\frac{x_{B(j)}^*}{\beta_{ji}} \leq \delta \leq \min_{j:\beta_{ji}<0} -\frac{x_{B(j)}^*}{\beta_{ji}}$$

- Otherwise, apply the dual simplex algorithm to reoptimize!

Local Sensitivity Analysis: Changing Cost Vector

Change c to $c + \delta \cdot e_j$, that is, only c_j is changed to $c_j + \delta$.

Observations:

- Feasibility is not affected but optimality – apply the primal simplex algorithm to reoptimize!
- Case 1: x_j is non-basic $\implies$ only $\bar{c}_j$ affected:

$$\hat{c}_j := c_j + \delta - {c_B}^T \cdot B^{-1} \cdot A_j = \bar{c}_j + \delta$$

  Thus, B remains optimal, if and only if $\delta \geq -\bar{c}_j$.

- Case 2: $x_j = x_{B(\ell)}$ is basic $\implies$ all reduced costs affected:

$$\begin{aligned} & c_i - (c_B + \delta \cdot e_\ell)^T \cdot B^{-1} \cdot A_i \geq 0 && \text{for all } i \neq j \\ \iff \quad & \bar{c}_i - \delta \cdot q_{\ell i} \geq 0 && \text{for all } i \neq j \end{aligned}$$

  where $q_{\ell i} = \ell$th entry of $B^{-1} \cdot A_i$.

Local Sensitivity Analysis: Changing Column of A

Change A_j to $A_j + \delta \cdot e_i$, that is, only a_{ij} is changed to $a_{ij} + \delta$.

Case 1: A_j is a non-basic column.

- B is still feasible but $\bar{c}_j$ is affected.
- If $c_j - \underbrace{p^T}_{{c_B}^T \cdot B^{-1}} \cdot (A_j + \delta \cdot e_i) = \bar{c}_j - \delta \cdot p_i \geq 0$, then B remains optimal.
- Otherwise, apply the primal simplex algorithm to reoptimize!

Case 2: A_j is a basic column: See exercises.

Global Dependence on the Right-Hand Side

$$P(b) := \{x \mid A \cdot x = b,\ x \geq 0\}$$
$$S := \{b \mid P(b) \neq \emptyset\} = \{A \cdot x \mid x \geq 0\} \qquad \text{(convex)}$$
$$F(b) := \min_{x \in P(b)} c^T \cdot x \qquad \text{for } b \in S$$

Assume that the dual feasible set is non-empty: $\{p \mid p^T \cdot A \leq c^T\} \neq \emptyset$

$$\Longrightarrow \qquad F(b) > -\infty \qquad \text{for all } b \in S.$$

Consider fixed $b^* \in S$ and assume that B is non-degenerate optimal basis:

$$x_B = B^{-1} \cdot b^* > 0 \ , \qquad \bar{c}^T = c^T - {c_B}^T \cdot B^{-1} \cdot A \geq 0 \ .$$

Changing b^* to b with $b - b^*$ sufficiently small, $B^{-1} \cdot b$ remains non-negative and B is still optimal.

$$\Longrightarrow \qquad F(b) = {c_B}^T \cdot B^{-1} \cdot b = p^T \cdot b \qquad \text{for } b \text{ close to } b^*.$$

Theorem 10.1.

$F(b)$ is a convex function of b. It is linear in vicinity of b^* with gradient p.

Proof: . . . □

Set of all Dual Optimal Solutions

Definition 10.2.

Let $S \subseteq \mathbb{R}^n$ convex, $F : S \to \mathbb{R}$ convex, and $b^* \in S$. Then $p \in \mathbb{R}^n$ is a subgradient of F at b^* if

$$F(b^*) + p^T \cdot (b - b^*) \leq F(b) \qquad \text{for all } b \in S.$$

Remarks:

- If F is linear in the vicinity of b^*, then there is a unique subgradient.
- If b^* is a breakpoint of F, then there are several subgradients.

Let $F(b) := \min\{c^T \cdot x \mid A \cdot x = b,\ x \geq 0\}$.

Theorem 10.3.

Suppose that the LP $\min c^T \cdot x$, s.t. $A \cdot x = b^*$, $x \geq 0$ is feasible and bounded. Then p is an optimal solution to the dual LP if and only if p is a subgradient of F at b^*.

Proof: . . . □

Global Dependence on the Cost Vector

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ with $\{x \mid A \cdot x = b,\ x \geq 0\} \neq \emptyset$ and for $c \in \mathbb{R}^n$

$$G(c) := \min\{c^T \cdot x \mid A \cdot x = b,\ x \geq 0\}$$
$$Q(c) := \{p \in \mathbb{R}^m \mid p^T \cdot A \leq c^T\}$$
$$T := \{c \in \mathbb{R}^n \mid Q(c) \neq \emptyset\} \qquad (convex).$$

Note that $T = \{c \mid G(c) > -\infty\}$ and, for $c \in T$,

$G(c) = \min_{i=1,\ldots,N} c^T \cdot x^i$ where $x^1, \ldots, x^N$ are the basic feasible solutions.

Theorem 10.4.

Consider a feasible linear program in standard form.

i. The set T is convex.
ii. $G(c)$ is a concave function on T.
iii. If, for some $c \in T$, the primal LP has a unique optimal solution x^*, then G is linear in the vicinity of c and its gradient is equal to x^*. □