Characterization of Unbounded LPs

Theorem 9.8.

Let $C := \{x \in \mathbb{R}^n \mid {a_i}^T \cdot x \geq 0,\ i = 1, \dots, m\}$ a pointed polyhedral cone and $c \in \mathbb{R}^n$. The minimal cost $c^T \cdot x$ subject to $x \in C$ is equal to $-\infty$ if and only if there is an extreme ray d of C with $c^T \cdot d < 0$.

Proof: . . . □

The result also holds if C is a polyhedron with at least one extreme point.

Theorem 9.9.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron with at least one extreme point and $c \in \mathbb{R}^n$. The minimal cost $c^T \cdot x$ subject to $x \in P$ is equal to $-\infty$ if and only if there is an extreme ray d of P with $c^T \cdot d < 0$.

Proof: . . . □

Remark: If the simplex method observes that an LP is unbounded, the corresponding jth basic direction is an extreme ray d with $c^T \cdot d < 0$.

Resolution Theorem

Theorem 9.10.

Let $P := \{x \in \mathbb{R}^n \mid A \cdot x \geq b\} \neq \emptyset$ pointed. Let $x^1, \dots, x^k$ be the extreme points and $w^1, \dots, w^r$ a complete set of extreme rays of P. Then,

$$P = \left\{ \sum_{i=1}^{k} \lambda_i \cdot x^i + \sum_{j=1}^{r} \theta_j \cdot w^j \mid \lambda_i, \theta_j \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1 \right\} .$$

Proof: . . . □

Corollary 9.11.

A non-empty polytope is equal to the convex hull of its extreme points. □

Corollary 9.12.

Every element of a pointed polyhedral cone is a non-negative linear combination of its extreme rays. □

Converse to the Resolution Theorem

Definition 9.13.

A set $Q \subseteq \mathbb{R}^n$ is finitely generated if there are $x^1, \dots, x^k, w^1, \dots, w^r \in \mathbb{R}^n$ such that

$$Q = \left\{ \sum_{i=1}^{k} \lambda_i \cdot x^i + \sum_{j=1}^{r} \theta_j \cdot w^j \mid \lambda_i, \theta_j \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\} .$$

Remark: The Resolution Theorem states that a polyhedron with at least one extreme point is finitely generated (also true for general polyhedra).

Theorem 9.14.

A finitely generated set is a polyhedron. In particular, the convex hull of finitely many vectors is a polytope.

Proof: . . . □

Representation of Polyhedra

Conclusion: There are two ways of representing a polyhedron:

i in terms of a finite set of linear constraints (outer representation);

ii as a finitely generated set, in terms of its extreme points and rays (inner representation).

Remarks:

- Passing from one type of description to the other is, in general, a complicated computational task.
- One description can be small while the other one is huge. Examples:
  - An n-dimensional cube is given by $2n$ linear constraints and has 2^n extreme points.
  - A representation of the convex hull of the $2n$ points

    $$e_1, -e_1, e_2, -e_2, \dots, e_n, -e_n$$

    in terms of linear constraints needs at least 2^n linear inequalities.

Chapter 10: Sensitivity Analysis for Linear Programs

(cp. Bertsimas & Tsitsiklis, Chapter 5)

Local Sensitivity Analysis

Consider a primal-dual pair of linear programs:

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \max \quad & p^T \cdot b \\ \text{s.t.} \quad & p^T \cdot A \leq c^T \end{aligned}$$

Let B be an optimal basis for the primal LP, i. e.,

$$B^{-1} \cdot b \geq 0 \qquad \text{(feasibility)},$$
$$c^T - {c_B}^T \cdot B^{-1} \cdot A \geq 0 \qquad \text{(optimality)},$$

and let x^* be a corresponding optimal basic solution.

Questions:

- Under what conditions does B remain feasible and optimal when problem data is being changed?
- What if B is no longer feasible or optimal after the change?

Local Sensitivity Analysis: Adding New Variable

Add a new variable x_{n+1} to the primal LP:

$$\begin{aligned} \min \;\; & c^T \cdot x + c_{n+1} \cdot x_{n+1} \\ \text{s.t.} \;\; & A \cdot x + A_{n+1} \cdot x_{n+1} = b \\ & (x, x_{n+1}) \geq 0 \end{aligned}$$

Observations:

- $(x^*, 0)$ is a basic feasible solution to the new problem.
- B remains optimal if $\bar{c}_{n+1} := c_{n+1} - {c_B}^T \cdot B^{-1} \cdot A_{n+1} \geq 0$.
- Otherwise apply the primal simplex algorithm to reoptimize!

Local Sensitivity Analysis: Adding New Inequality

Add a new inequality ${a_{m+1}}^T \cdot x \geq b_{m+1}$ to the primal LP.

Observations:

- If x^* satisfies the new constraint, it remains optimal.
- Otherwise, introduce slack variable $x_{n+1} \geq 0$ and rewrite:

$${a_{m+1}}^T \cdot x - x_{n+1} = b_{m+1}$$

- Matrix A is replaced by $\bar{A} := \begin{pmatrix} A & 0 \\ {a_{m+1}}^T & -1 \end{pmatrix}$.
- New basis $\bar{B} := \begin{pmatrix} B & 0 \\ {a_{\dots}}^T & -1 \end{pmatrix}$ with $\bar{B}^{-1} = \begin{pmatrix} B^{-1} & 0 \\ {a_{\dots}}^T \cdot B^{-1} & -1 \end{pmatrix}$ and associated basic solution $(x^*, {a_{m+1}}^T \cdot x^* - b_{m+1})$.
- New reduced cost vector

$$(c^T, 0) - ({c_B}^T, 0) \cdot \bar{B}^{-1} \cdot \bar{A} = (c^T - {c_B}^T \cdot B^{-1} \cdot A, 0) \geq 0 \;\; .$$

- Apply the dual simplex algorithm to reoptimize!