Discrete Optimization (ADM II)

Britta Peis & Martin Skutella

General Remarks

- new flavor of ADM II — continuation of "new ADM I" from last semester
- lectures (Britta Peis & Martin Skutella):
  Tuesday, 12:15 – 13:45, MA 041
  Friday, 10:15 – 11:45, MA 042
- exercise session (Britta Peis & Martin Skutella):
  Thursday, 12:15 – 13:45, MA 650 (starting next week)
- tutorial sessions (Ágnes Cseh)
  Tuesday, 14 – 16, MA 850 (starting next week)
  Tuesday, 16 – 18, MA 548 (starting next week)
- homework: set of problems every week (solve in groups of at most 3)
- "Scheinkriterium": 50 % of points from problem sets 1–6 and 7–12
- final oral exam (Modulabschlussprüfung) in summer during the semester break (details t.b.a.)

Outline

- Advanced Linear Programming & Polyhedral Theory
- Matchings
- Integrality of Polyhedra & Integer Programming
- Traveling Salesperson Problem
- Matroids
- Approximation Algorithms

Chapter 9: Linear Programming & Polyhedral Theory

(cp. Bertsimas & Tsitsiklis, Chapters 2.8, 4.6–4.9)

Fourier-Motzkin Elimination

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. If follows from Gaussian elimination that

$$\exists x \in \mathbb{R}^n : A \cdot x = b \iff \nexists y \in \mathbb{R}^m : y^T \cdot A = 0^T,\ y^T \cdot b = -1$$

Aim: Derive analogous characterization for existence of x with $A \cdot x \leq b$.

Idea: Rewrite $A \cdot x \leq b$ equivalently (multiply rows by positive scalars) as

$$\begin{aligned} x_1 + (a'_i)^T \cdot x' &\leq b_i && \text{for all } i = 1, \dots, m' \\ -x_1 + (a'_i)^T \cdot x' &\leq b_i && \text{for all } i = m'+1, \dots, m'' \\ (a'_i)^T \cdot x' &\leq b_i && \text{for all } i = m''+1, \dots, m \end{aligned}$$

with $x' = (x_2, \dots, x_n)^T$ and $(a'_i)^T = i$th row of A with first entry deleted.

This system has a solution (x_1, x') if and only if there is $x' \in \mathbb{R}^{n-1}$ with

$$\begin{aligned} (a'_j)^T \cdot x' - b_j \leq b_i - (a'_i)^T \cdot x' && \text{for all } i = 1, \dots, m',\ j = m'+1, \dots, m'' \\ (a'_i)^T \cdot x' \leq b_i && \text{for all } i = m''+1, \dots, m \end{aligned}$$

Farkas' Lemma

The system $A \cdot x \leq b$ has a solution x if and only if there is $x' \in \mathbb{R}^{n-1}$ with

$$\begin{aligned} (a'_i + a'_j)^T \cdot x' \leq b_i + b_j && \text{for all } i = 1, \dots, m',\ j = m'+1, \dots, m'' \\ (a'_i)^T \cdot x' \leq b_i && \text{for all } i = m''+1, \dots, m \end{aligned}$$

The following theorem is known as Farkas' Lemma:

Theorem 9.1.

The system $A \cdot x \leq b$ has a solution x, if and only if there is no vector y satisfying $y \geq 0$, $y^T \cdot A = 0$ and $y^T \cdot b < 0$.

Proof: ... □

Corollary 9.2.

The system $A \cdot x = b$ has a non-negative solution $x \geq 0$, if and only if there is no vector y satisfying $y^T \cdot A \geq 0$ and $y^T \cdot b < 0$. □

Remark: Farkas' Lemma also follows immediately from LP duality.

Separating Hyperplane Theorem

Another possible approach to Farkas' Lemma is the following separating hyperplane theorem:

Theorem 9.3.

Let S be a non-empty, closed, convex subset of $\mathbb{R}^n$ and let $x^* \in \mathbb{R}^n$ be a vector that does not belong to S. Then there exists some vector $c \in \mathbb{R}^n$ such that $c^T \cdot x^* < c^T \cdot x$ for all $x \in S$.

Proof: ... □

Cones

Definition 9.4.

a. A set $C \subseteq \mathbb{R}^n$ is a cone if $\lambda \cdot x \in C$ for all $\lambda \geq 0$ and all $x \in C$.
b. A polyhedron of the form $P = \{x \in \mathbb{R}^n \mid A \cdot x \geq 0\}$ is called polyhedral cone.

Remark: $0 \in P$ is the only possible vertex of a polyhedral cone P.
If $0 \in P$ is a vertex, then P is called pointed.

Theorem 9.5.

Let $C \subseteq \mathbb{R}^n$ be the polyhedral cone defined by the constraints ${a_i}^T \cdot x \geq 0$, $i = 1, \dots, m$. Then, the following are equivalent:

i. The zero vector is an extreme point of C.
ii. The cone C does not contain a line.
iii. There exist n vectors out of the family $a_1, \dots, a_m$, which are linearly independent.

Proof: Follows from Theorem 2.23. □

Recession Cones

Definition 9.6.

Consider a non-empty polyhedron $P = \{x \in \mathbb{R}^n \mid A \cdot x \geq b\}$ and $y \in P$. The recession cone of P (at y) is the set

$$\{d \in \mathbb{R}^n \mid y + \lambda \cdot d \in P \text{ for all } \lambda \geq 0\} \ .$$

The non-zero elements of the recession cone are the rays of P.

Remarks:

- Notice that

$$\begin{aligned}&\{d \in \mathbb{R}^n \mid y + \lambda \cdot d \in P \text{ for all } \lambda \geq 0\}\\ &\quad = \{d \in \mathbb{R}^n \mid A \cdot (y + \lambda \cdot d) \geq b \text{ for all } \lambda \geq 0\}\\ &\quad = \{d \in \mathbb{R}^n \mid A \cdot d \geq 0\} \ .\end{aligned}$$

- The definition of the recession cone of P is independent of $y \in P$.
- The recession cone of P is a polyhedral cone.
- The recession cone of $P = \{x \in \mathbb{R}^n \mid A \cdot x = b, \ x \geq 0\}$ is

$$\{d \in \mathbb{R}^n \mid A \cdot d = 0, \ d \geq 0\} \ .$$

Extreme Rays

Observation.

A non-empty polyhedron P has a vertex if and only if its recession cone is pointed. In this case we also say that P is pointed.

Definition 9.7.

a. A non-zero element x of a polyhedral cone $C \subseteq \mathbb{R}^n$ is an extreme ray if there are $n - 1$ linearly independent constraints that are active at x.
b. An extreme ray of the recession cone of a polyhedron P is also called an extreme ray of P.

Remark: Up to multiplication with positive factors, there are only finitely many extreme rays of a polyhedron.