Christofides' Heuristic (cont.')

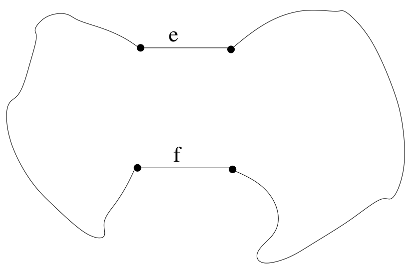
Theorem 14.2.
For the Euclidean TSP with nonnegative costs, Christofides' Heuristic produces a tour of cost at most $\frac{3}{2}$ times the cost of an optimal tour.

Proof: . . . □

Christofides' on TSPLIB:

- Christofides tours are about 1.14 times the optimum.
- By always choosing the best "shortcut" for the given node v, the performance of the algorithm improves to 1.09 times the optimum.

Tour Improvements Methods: 2-opt and k-opt



Note: Given any tour T, if two nonadjacent distinct edges $e, f \in T$ are deleted from T, there is a unique way to choose $\{g, h\} \subseteq E \setminus \{e, f\}$ in order to combine the two resulting paths to a tour

$$T' = T \setminus \{e, f\} \cup \{g, h\}.$$

This is called 2-interchange.

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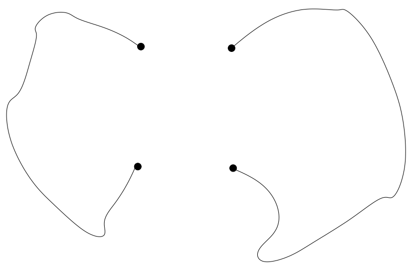

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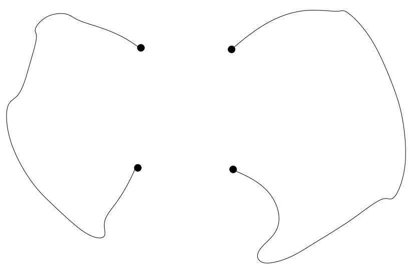

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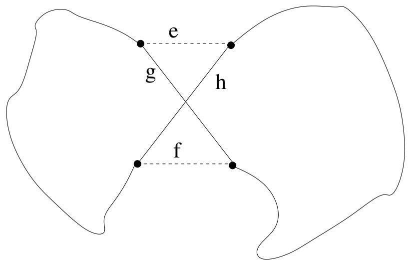

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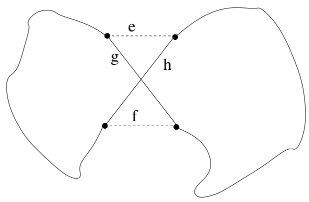
2-Opt and k-Opt



2-opt method:

- While there exists a tour $T' = T \setminus \{e, f\} \cup \{g, h\}$ with two nonadjacent $e, f \in T$ and $\{g, h\} \neq \{e, f\}$ such that $c(T') < c(T)$,
- set $T = T'$;

A tour T is called 2-optimal if no such tour T' with $c(T') < c(T)$ exists.

Remark on runtime: Each iteration takes $\mathcal{O}(|V|^2)$. But: Might need an exponential number of iterations before a 2-optimal tour is found!

Generalization to k-opt methods: check whether any subset of k nonadjacent edges can be replaced by a better choice of k edges.

On TSPLIB: 2-opt [3-opt] produced tours are about 1.06 [1.04] times the optimum

Lower Bounds

Let $G = (V, E)$ with $c \in \mathbb{R}^E$ be an instance of a TSP, and let $\mathcal{T}$ be the set of all tours.Then $B \in \mathbb{R}$ is a lower bound on the TSP if $c(T) \geq B$ for each $T \in \mathcal{T}$.

Lower bounds can be used to

- prove the optimality of a given solution/tour, or
- measure the quality of a given solution/tour.

Example: ...

1-tree bound: Given some fixed node $v_1 \in V$, let $A := \min\{c_e + c_f \mid e, f \in \delta(v_1),\ e \neq f\}$, and let B be the cost of a MST on $G \setminus \{v_1\}$. Then $A + B$ is a lower bound on the TSP, called 1-tree bound.

Linear Programming

Note: The characteristic vector of a tour satisfies

$$\begin{aligned} x(\delta(v)) &= 2 \quad \forall v \in V \\ 0 \leq x_e &\leq 1 \quad \forall e \in E \end{aligned}$$

However, not every integer solution is a tour. Example: ...

The subtour (elimination) constraints

$$x(\delta(S)) \geq 2 \quad \forall S \subseteq V,\ \emptyset \neq S \neq V$$

forbid these subtours (small circuits).

Linear Relaxation of TSP

Dantzig, Fulkerson, and Johnson's relaxation of the TSP [1954]:

$$
\begin{aligned}
z^*(G) = \min \quad & \textstyle\sum_{e \in E} c_e x_e & & & \\
\text{s.t.} \quad & x(\delta(v)) & = & \; 2 & \forall v \in V \\
& x(\delta(S)) & \geq & \; 2 & \forall S \subset V, S \neq \emptyset \\
& 0 \leq & x_e & \leq 1 & \forall e \in E
\end{aligned}
\qquad (14.1)
$$

Note: Any integral solution is a tour.

Thus, $z^*(G)$ is a lower bound, called the subtour bound.

But: Exponentially many subtour constraints! Even if we restrict to sets $S \subset V$ with $|S| \leq \frac{|V|}{2}$.

Moreover, way too many variables!

Cutting Planes

Cutting-plane approach to solve the linear relaxation 14.1:

- Compute an optimal solution x^* of $\min\{c^T x \mid x(\delta(v)) = 2, \forall v \in V, 0 \leq x \leq 1\}$.
- STOP, if x^* is the characteristic vector of a tour.
- Else, try to find a subtour constraint violated by x^*; add it to the system, and iterate.

Note: If no violated subtour constraint, x^* is an optimal solution of 14.1.
Questions:

1. Efficient way to find a violated subtour constraint, or prove that none exists?
2. Efficient way to solve the LP in each iteration?

Idea for Q2: Handle the huge set of variables in a similiar way we handle the cutting planes:

- Start out with an LP on a subset of variables.
- Add in the remaining ones as they are needed.

Column Generation

Column generation to solve 14.1:

1. Select a set $E' \subseteq E$ such that the LP restricted to E' has a feasible solution. (E.g., take the union of 10 tours given by some heuristic.)
2. Compute an optimal solution x' of this LP, and an optimal solution (y', Y') of the corresponding dual problem.
3. Extend x' to a feasible solution x^* of 14.1 by setting $x^*_e = \max\{x'_e, 0\}$.
4. If (y', Y') is feasible to the dual of 14.1, then x^* is an optimal solution to 14.1.
5. Otherwise, add those edges $e \in E \setminus E'$ to E' for which the dual constraint is violated, and iterate.

This is, how we can answer Q2, but how about Q1?

Detecting Violated Subtour Constraints

Q1: Efficient way to find violated subtour constraints?

1. Compute an optimal solution x^* of $\min\{c^T x \mid x(\delta(v)) = 2, \forall v \in V, 0 \leq x \leq 1\}$.
2. Let $E^* = \{e \in E \mid x^*_e > 0\}$.
3. WHILE $G^* = (V, E^*)$ is disconnected, add the constraint $x(\delta(S)) \geq 2$ for each $S \subseteq V$ that forms a maximal connected component;
4. Compute the minimum cut $S^* = \text{argmin}\{u(\delta(S)) \mid \emptyset \neq S \subset V\}$ in G w.r.t. capacities $u_e = x^*_e$.
5. If $u(\delta(S^*)) < 2$, add the constraint $x(\delta(S^*)) \geq 2$ to the LP and iterate;
6. Otherwise STOP; $(u(\delta(S^*)) \geq 2$ proves that x^* is an optimal (fractional) solution to 14.1).

Additional Cutting Planes?

Combining column generation with the cutting-plane approach allows us to solve LPs that are both, "long" and "wide".

Now, given an optimal solution of the LP 14.1, this is in general NOT the characteristic vector of a tour. What next?

- Either, we could stop with a (pretty good) lower bound and go on to branch-and-bound (see below),
- or, we could try to find some other classes of cutting planes to add, and continue with our cutting-plane algorithm.

Example: ...

Comb Inequalities

Definition 14.3.

A comb consists of a nonempty handle $H \subseteq V$, $H \neq V$ and $2k+1$ pairwise disjoint, nonempty teeth $T_1, T_2, \ldots, T_{k+1} \subseteq V$ for some $k \geq 1$ such that $T_i \cap H \neq \emptyset \neq T_i \cap (V \setminus H)$ for each $i = 1, \ldots, 2k+1$.

Theorem 14.4 (Chvátal'73, Grötschel & Padberg'79).

Let C be a comb with handle H and teeth $T_1, T_2, \ldots, T_{2k+1}$ for $k \geq 1$. Then the characteristic vector x of any tour satisfies

$$x(\gamma(H)) + \sum_{i=1}^{2k+1} (|T_i| - 1) - (k+1).$$

Proof: ... □