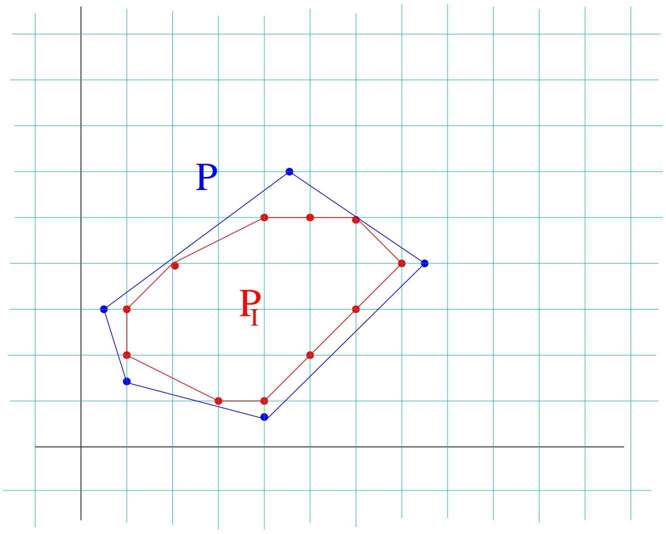
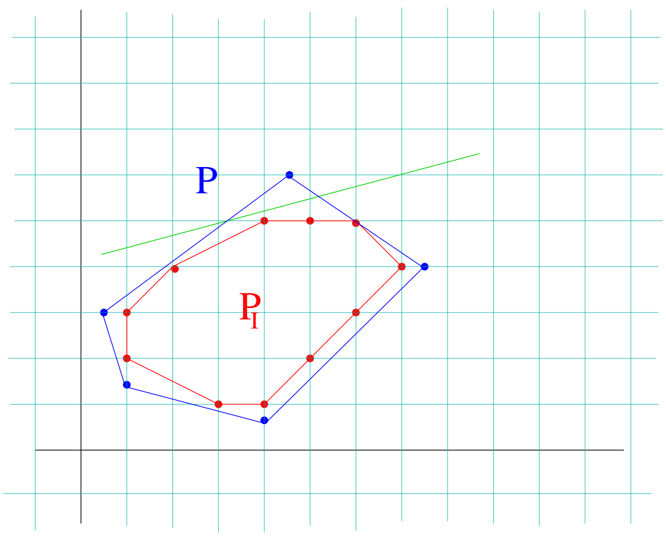
Cutting-Plane Proofs

A cutting-plane proof of $w^T x \leq t$ from system $\{a_i^T x \leq b_i \mid i = 1, \ldots m\}$ is a sequence of inequalities $a_{m+k}^T x \leq b_{m+k}$ for $k = 1, \ldots, \tilde{m}$ together with values $y_{kj} \in \mathbb{R}_+$ $(1 \leq k \leq \tilde{m};\ 1 \leq j \leq m + k - 1)$ such that each $a_{m+k}^T x \leq b_{m+k}$ is derived from $\{a_i^T x \leq b_i \mid i = 1, \ldots, m + k - 1\}$ using the values y_{kj} and such that $w^T x \leq t$ is the last inequality in the sequence.



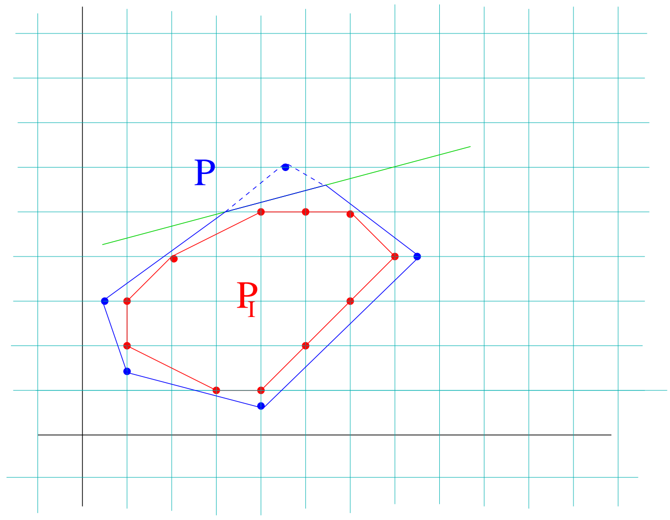
Cutting-Plane Proofs

A cutting-plane proof of $w^T x \leq t$ from system $\{a_i^T x \leq b_i \mid i = 1, \ldots m\}$ is a sequence of inequalities $a_{m+k}^T x \leq b_{m+k}$ for $k = 1, \ldots, \tilde{m}$ together with values $y_{kj} \in \mathbb{R}_+$ $(1 \leq k \leq \tilde{m};\ 1 \leq j \leq m + k - 1)$ such that each $a_{m+k}^T x \leq b_{m+k}$ is derived from $\{a_i^T x \leq b_i \mid i = 1, \ldots, m + k - 1\}$ using the values y_{kj} and such that $w^T x \leq t$ is the last inequality in the sequence.



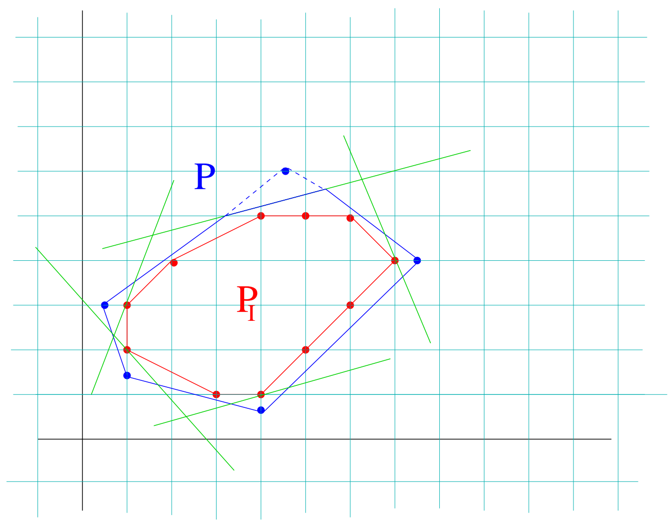
Cutting-Plane Proofs

A cutting-plane proof of $w^T x \leq t$ from system $\{a_i^T x \leq b_i \mid i = 1, \dots m\}$ is a sequence of inequalities $a_{m+k}^T x \leq b_{m+k}$ for $k = 1, \dots, \tilde{m}$ together with values $y_{kj} \in \mathbb{R}_+$ $(1 \leq k \leq \tilde{m};\ 1 \leq j \leq m + k - 1)$ such that each $a_{m+k}^T x \leq b_{m+k}$ is derived from $\{a_i^T x \leq b_i \mid i = 1, \dots, m + k - 1\}$ using the values y_{kj} and such that $w^T x \leq t$ is the last inequality in the sequence.



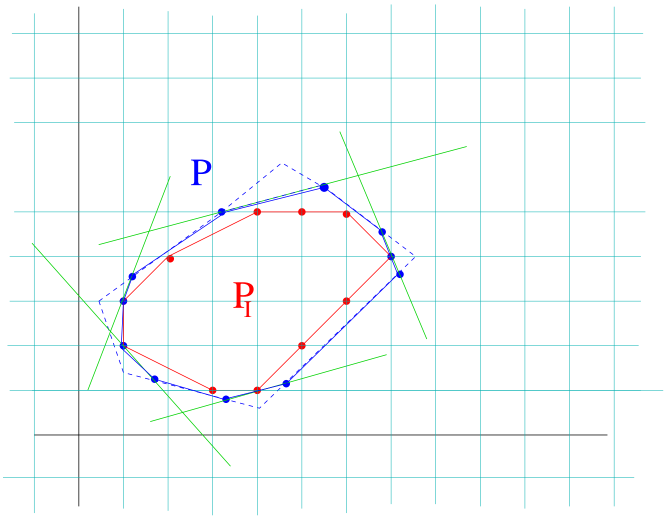
Cutting-Plane Proofs

A cutting-plane proof of $w^T x \leq t$ from system $\{a_i^T x \leq b_i \mid i = 1, \dots m\}$ is a sequence of inequalities $a_{m+k}^T x \leq b_{m+k}$ for $k = 1, \dots, \tilde{m}$ together with values $y_{kj} \in \mathbb{R}_+$ $(1 \leq k \leq \tilde{m};\ 1 \leq j \leq m + k - 1)$ such that each $a_{m+k}^T x \leq b_{m+k}$ is derived from $\{a_i^T x \leq b_i \mid i = 1, \dots, m + k - 1\}$ using the values y_{kj} and such that $w^T x \leq t$ is the last inequality in the sequence.



Cutting-Plane Proofs

A cutting-plane proof of $w^T x \le t$ from system $\{a_i^T x \le b_i \mid i = 1, \dots m\}$ is a sequence of inequalities $a_{m+k}^T x \le b_{m+k}$ for $k = 1, \dots, \tilde{m}$ together with values $y_{kj} \in \mathbb{R}_+$ $(1 \le k \le \tilde{m};\ 1 \le j \le m + k - 1)$ such that each $a_{m+k}^T x \le b_{m+k}$ is derived from $\{a_i^T x \le b_i \mid i = 1, \dots, m + k - 1\}$ using the values y_{kj} and such that $w^T x \le t$ is the last inequality in the sequence.



Cutting-plane proofs always exist

Theorem 14.34.

Let $P = \{x \mid Ax \le b\}$ be a rational polytope and let $w^T x \le t$ be a cutting plane for P with w integral. Then there exists a cutting-plane proof of $w^T x \le t'$ from $Ax \le b$, for some $t' \le t$.

A special case occurs when P contains no integral vector:

Theorem 14.35.

Let $P = \{x \mid Ax \le b\}$ be a rational polytope that contains no integral vector. Then there exists a cutting-plane proof of $0^T x \le -1$ from $Ax \le b$.

Analogue version of Farkas' Lemma: Polyhedron $P = \{x \mid Ax \le b\}$ is empty$\iff$ $0^T x \le -1$ can be written as a nonnegative linear combination of the inequalities in $Ax \le b$. (Exercise).

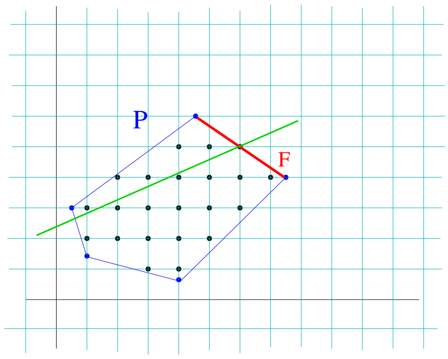
Note: $w^T x \le t$ is a cut for P $\iff$ $P \cap \{x \mid w^T x \ge \lfloor t \rfloor + 1\}$ contains no integral vector.

Proofs of Theorems 14.35 and 14.34

In order to proof the Theorems we need

Lemma 14.36.

Let F be a face of a rational polytope P. If $c^T x \leq \lfloor d \rfloor$ is a GC-cutting-plane for F, then there exists a GC-cutting-plane $\tilde{c}^T x \leq \lfloor \tilde{d} \rfloor$ for P such that $F \cap \{x \mid \tilde{c}^T x \leq \lfloor \tilde{d} \rfloor\} = F \cap \{x \mid c^T x \leq \lfloor d \rfloor\}$.



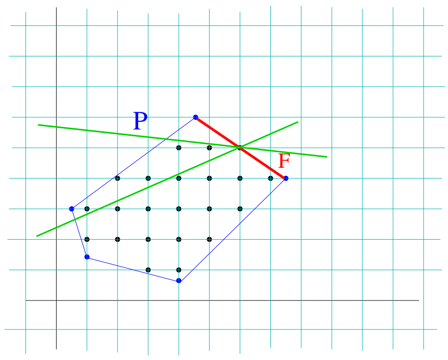
Proofs of Lemma, Theorem 14.34 and Theorem 14.35:... □

Proofs of Theorems 14.35 and 14.34

In order to proof the Theorems we need

Lemma 14.36.

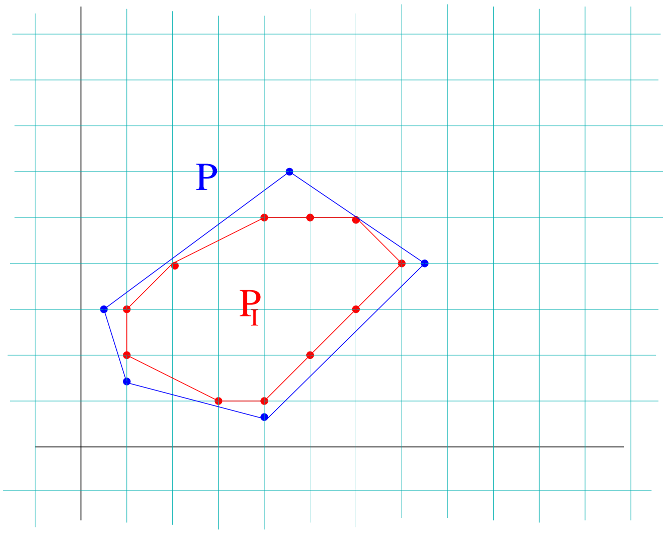
Let F be a face of a rational polytope P. If $c^T x \leq \lfloor d \rfloor$ is a GC-cutting-plane for F, then there exists a GC-cutting-plane $\tilde{c}^T x \leq \lfloor \tilde{d} \rfloor$ for P such that $F \cap \{x \mid \tilde{c}^T x \leq \lfloor \tilde{d} \rfloor\} = F \cap \{x \mid c^T x \leq \lfloor d \rfloor\}$.



Proofs of Lemma, Theorem 14.34 and Theorem 14.35:... □

Procedure to find a linear description of P_I.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron, and let P_I :=conv.hull($P \cap \mathbb{Z}^n$) be the convex hull of the integral points in P.

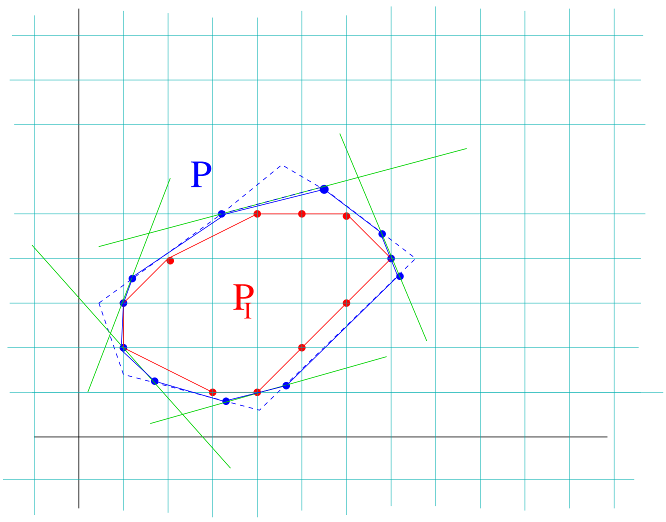


Note: Adding proper cuts, i.e., cuts that "really" cut off parts of $P \setminus P_I$ from P, provides successively tighter approximations to P_I.

$\Longrightarrow$ procedure to find a linear description of P_I.

Procedure to find a linear description of P_I.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron, and let P_I :=conv.hull($P \cap \mathbb{Z}^n$) be the convex hull of the integral points in P.



Note: Adding proper cuts, i.e., cuts that "really" cut off parts of $P \setminus P_I$ from P, provides successively tighter approximations to P_I.

$\Longrightarrow$ procedure to find a linear description of P_I.

Chvátal Rank

Procedure to obtain a linear description of P_I:

1. (i) Let $P^{(0)} = P$ and $i = 0$;
2. (ii) Add all possible Gomory-Chvátal cuts to $P^{(i)}$ to obtain
$$P^{(i+1)} = \{x \in P^{(i)} \mid x \text{ satisfies every GC-cut for } P^{(i)}\}.$$
3. (iii) Iterate with $i = i + 1$;

This gives a chain $P = P^{(0)} \supseteq P^{(1)} \supseteq \ldots \supseteq P_I$.

Theorem 14.37 (Schrijver 1980).

If $P^{(i)}$ is a rational polytope, then $P^{(i+1)}$ is also a rational polytope.

Proof:... □

Corollary 14.38.

$P^{(k)} = P_I$ for some integer k.

The least k for which $P^{(k)} = P_I$ is called Chvátal rank of P.

Cutting-Plane Algorithms

Let P be a polyhedron, and consider an ILP of type $\max\{w^T \mid x \in P,\ x \text{ integral}\}$. Given some classes of cut-inequalities, we may try the following procedure:

1. Find an optimal solution x^* of $\max\{w^T \mid x \in P\}$.
2. If x^* is integral, we are done.
3. Otherwise, try to find a cut-inequality violated by x^*, add it to P, and iterate.

If we are lucky, the cutting-plane algorithm terminates with an integral optimal solution, and thus solves the ILP.

But, off course, it might terminate at some point where we do not find a violated cut.

In any case: the optimal value of the LP-relaxation in each iteration provides an upper bound!

Useful in, for example, *branch-and-bound* (or *branch-and-cut*) algorithms.