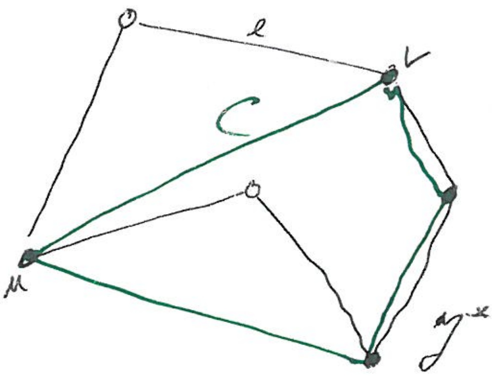
Theorem 15.2: If $c \geq 0$ and Δ-inequality is satisfied, any tour constructed by Christofides' Heuristic has cost at most $\frac{3}{2}$ times the cost of an optimal tour.

Proof: Let J^* be an optimal tour. Thus, for any $e \in J^*$ we have

$c(T) \leq c(J^* - e) \leq c(J^*)$.



Define a circuit C on W by joining the nodes in C in the order as they appear in J^*.
The edges in C partition into two perfect matchings of $G[W]$, say M_1 and M_2.

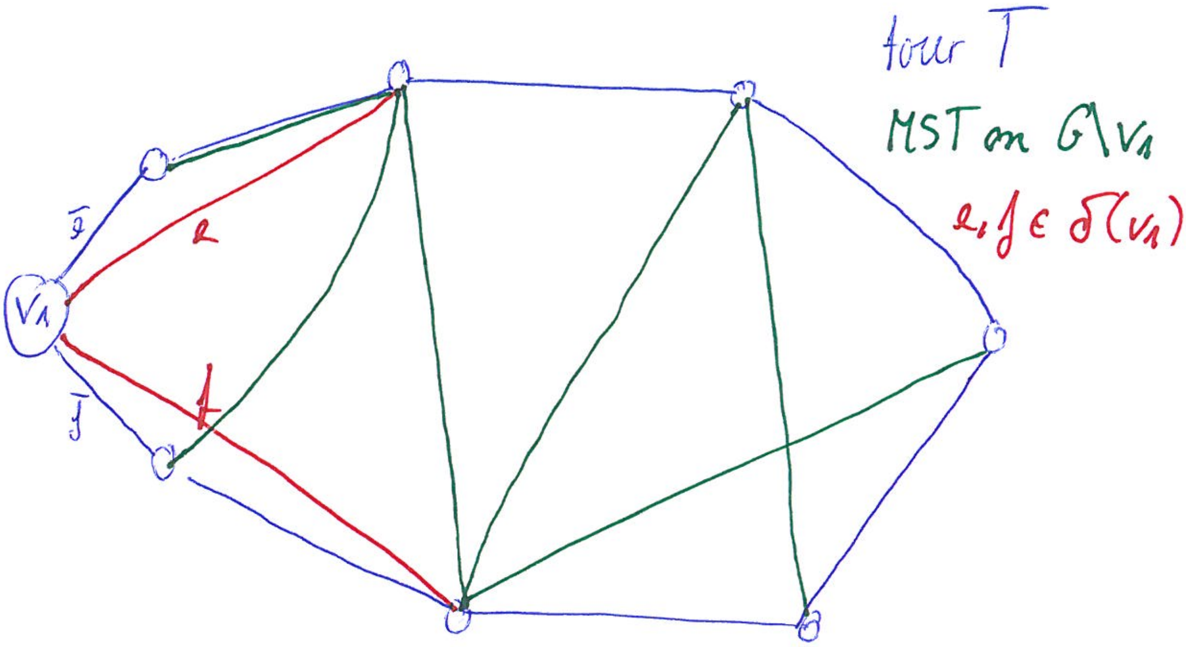
$$\Delta\text{-inequality} \Rightarrow \forall e \in C: \; c(\underset{\{u,v\}}{\underset{\parallel}{\hat{e}}}) \leq c(P_{J^*}[u,v])$$

$$\Rightarrow c(M) \leq \min_{i \in 1,2} c(M_i) \leq \frac{c(J^*)}{2}$$

$$\Rightarrow c(J) \leq \frac{3}{2} \cdot c(J^*)$$

And shortcutting can only improve $c(J)$. $\square$

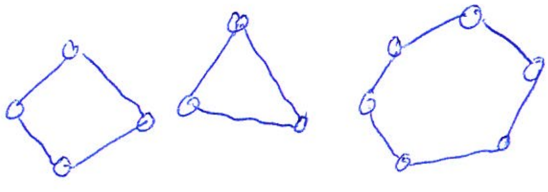
1-tree bound



$$\left.\begin{array}{l} c(T \setminus \{\bar{e}, \bar{f}\}) \geq c(MST(G\setminus v_1)) =: B \\ c_{\bar{e}} + c_{\bar{f}} \geq \min\{c_e + c_f \mid e \neq f \in \delta(v_1)\} =: A \end{array}\right\} \Rightarrow c(T) \geq A + B$$

2-factor

A <u>two-factor</u> is a disjoint union of circuits meeting all nodes.



$\rightsquigarrow$ characteristic vector of a two-factor is an integral solution of $x(\delta(v)) = 2 \;\forall v \in V$, $0 \leq x_e \leq 1 \;\forall e \in E$.

Column Generation to solve

$$(P') \quad \begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & x(\delta(v)) = 2 \quad \forall v \in V \\ & x(\delta(S)) \geq 2 \quad \forall \emptyset \neq S \subset V \\ & 0 \leq x_e \leq 1 \quad \forall e \in E' \end{aligned}$$

Dual Problem:

$$(D') \quad \begin{aligned} \max\ & \sum_{v \in V} 2 y_v + \sum_{\substack{S \subseteq V, \\ S \neq \emptyset, V}} 2 y_S \\ \text{s.t.}\ & y_u + y_v + \sum_{\substack{S \subseteq V \\ S \neq \emptyset, V}} \{ y_S : \{u,v\} \in \delta(S) \} \leq c_{uv} \quad \forall \{u,v\} \in E' \\ & y_S \geq 0 \quad \forall S \subseteq V, S \neq \emptyset, S \neq V \end{aligned}$$