Proof of Theorem 16.4:

(i) $\Rightarrow$ (ii) clear.

(ii) $\Rightarrow$ (iii) Assume that $B_1, B_2 \subseteq X \subseteq E$ are bases of different card., i.e.

$|B_1| > |B_2|$.

Take $Y \subseteq B_1$ with $|Y| = |B_2| + 1$

$$\overset{(ii)}{\Longrightarrow} \exists\, e \in Y \setminus B_2 : \underbrace{B_2 \cup \{e\}}_{\subseteq X} \in \mathcal{F} \quad \lightning\ B_2 \text{ basis of } X.$$

(iii) $\Rightarrow$ (i): Let $X, Y \in \mathcal{F}$, $|X| > |Y|$

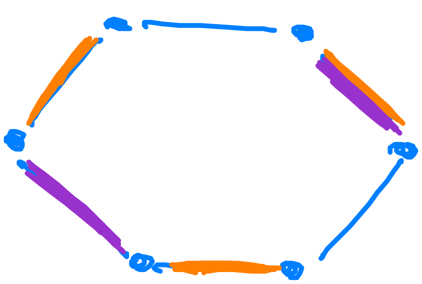
Since all bases of $X \cup Y$ have the same cardinality, Y cannot be a basis

$\Rightarrow \exists\, e \in X \setminus Y : Y \cup \{e\} \in \mathcal{F}$. $\square$

Example:

E = set of edges

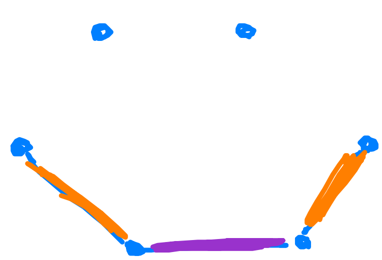
$\mathcal{F} = \{\text{matchings}\}$



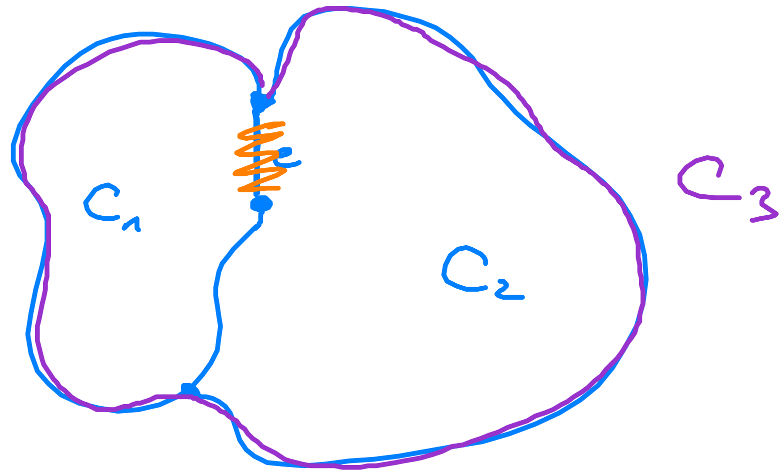
$r(E) = 3$

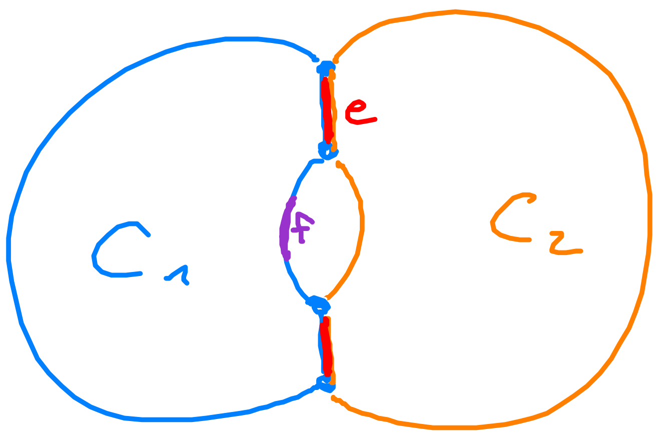
$\rho(E) = 2$

X:



rank quotient: $\frac{1}{2}$





Proof of Theorem 16.11:

Let $G_n \subseteq E$ be the greedy solution and $O_n \subseteq E$ an opt. solution.

Let $E_j := \{1, \dots, j\}$, $G_j := G_n \cap E_j$

$O_j := O_n \cap E_j$

for $j = 0, \dots, n$

$$
\begin{aligned}
c(G_n) &= \sum_{j=1}^{n} \left(|G_j| - |G_{j-1}|\right) \cdot c(e_j) \\
&= \sum_{j=1}^{n} |G_j| \cdot d_j \qquad \text{with } d_j := c(e_j) - c(e_{j+1}) \geq 0 \\
&\qquad\qquad\qquad\qquad\quad d_n := c(e_n) \\
&\geq \sum_{j=1}^{n} g(E_j) \cdot d_j \\
&\geq q(E, \mathcal{F}) \cdot \sum_{j=1}^{n} r(E_j) \cdot d_j \\
&\geq q(E, \mathcal{F}) \cdot \sum_{j=1}^{n} |\sigma_j| \cdot d_j \\
&= q(E, \mathcal{F}) \cdot \sum_{j=1}^{n} \left(|\sigma_j| - |\sigma_{j-1}|\right) \cdot c(e_j) \\
&= q(E, \mathcal{F}) \cdot c(\sigma_n)
\end{aligned}
$$

To prove the tightness of the lower bound, let $F \subseteq E$ with bases B_1, B_2 of F such that $\frac{|B_1|}{|B_2|} = q(E, \mathcal{F})$.

Let $c(e) := \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{otherwise} \end{cases}$

Sort E such that $c(e_1) \geq c(e_2) \geq \cdots \geq c(e_n)$ and $B_1 = \{e_1, \ldots, e_{|B_1|}\}$. Then,

$$G(E, \mathcal{F}, c) = |B_1| \text{ and } OPT(E, \mathcal{F}, c) = |B_2|$$

$\square$

Proof of Theorem 16.12:

"$\subseteq$" clear.

"$\supseteq$": Show that all vertices of the given polytope are integral.

Let $c \in \mathbb{R}^E$ and show that

$$\max\{c^T x \mid x \geq 0, \sum_{e \in X} x_e \leq r(X) \ \forall X \subseteq E\}$$

has integral optimum solution x.

W.l.o.g. $c \geq 0$

$$c(G_n) = \sum_{j=1}^{n} r(E_j) \cdot d_j \quad \text{(see above)}$$

$$\geq \sum_{j=1}^{n} \Big(\sum_{e \in E_j} x_e\Big) \cdot d_j$$

$$= \sum_{e \in E} c_e \cdot x_e = c^T x$$

$\square$