

$S$  is convex:

$$\zeta^1, \zeta^2 \in S, 0 \leq \lambda \leq 1$$

Prove that  $\lambda \cdot \zeta^1 + (1-\lambda) \cdot \zeta^2 \in S$

$$\text{Write } \zeta^1 = A \cdot x^1, x^1 \geq 0$$

$$\zeta^2 = A \cdot x^2, x^2 \geq 0$$

$$\text{Set } \bar{x} := \lambda \cdot x^1 + (1-\lambda) \cdot x^2 \geq 0$$

$$\begin{aligned} S \ni A \cdot \bar{x} &= \lambda \cdot A \cdot x^1 + (1-\lambda) \cdot A \cdot x^2 \\ &= \lambda \cdot \zeta^1 + (1-\lambda) \cdot \zeta^2 \quad \square \end{aligned}$$

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Proof of Theorem 10.1:

Second part (linearity and gradient) clear.

First part: Consider the feasible dual program:

$$\max p^T \cdot b$$

$$\text{s.t. } p^T A \leq c^T$$

$$\text{rank}(A) = m$$

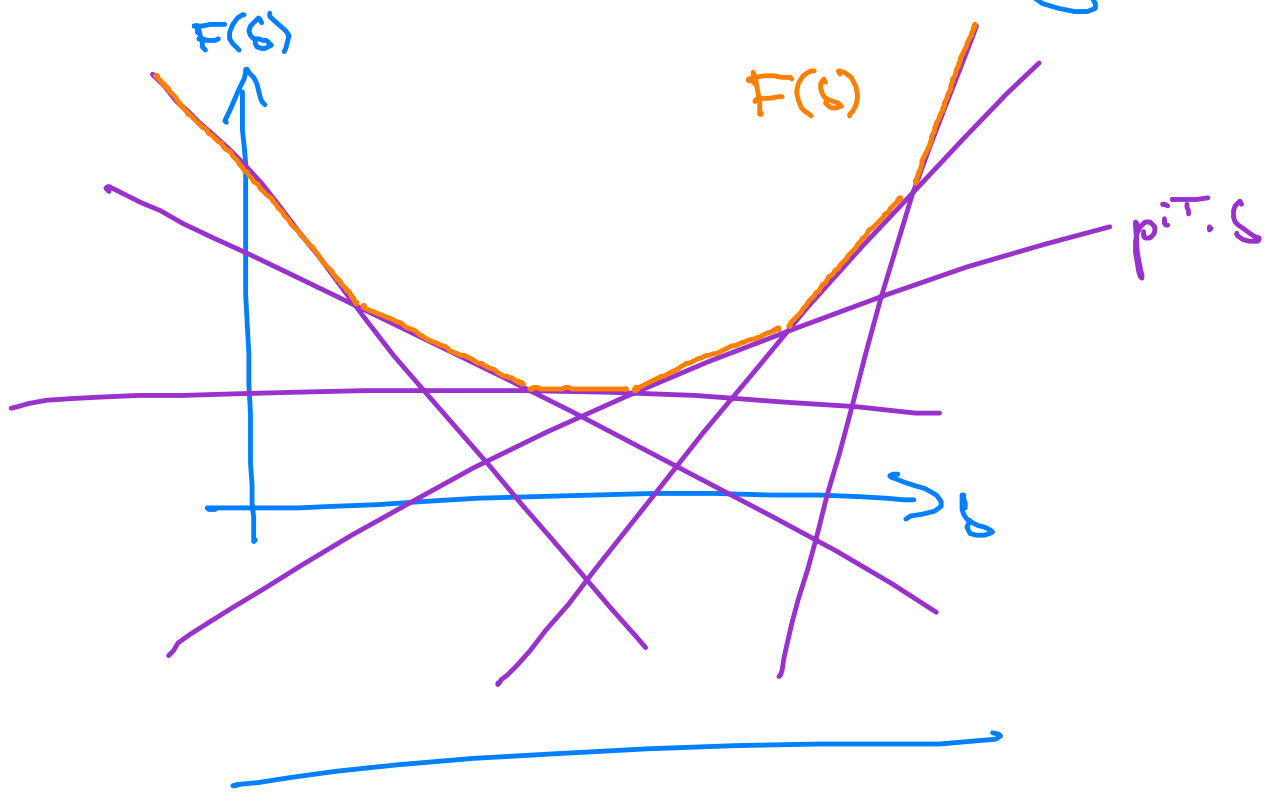
$\Rightarrow$  extreme points do exist

Let  $p^1, \dots, p^N$  be the extreme points of  $\{p \mid p^T A \leq c^T\}$ . By strong duality:

$$F(b) = \max_{i=1, \dots, N} p_i^T \cdot b \quad \text{for } b \in S$$

Notice that a linear function is convex

and the maximum of convex functions is a convex function again.  $\square$



### Proof of Theorem 10.3:

$\Leftarrow$ : Let  $p$  be an opt. dual solution

$$\Rightarrow p^T \cdot b^* = F(s^*). \text{ For } s \in S \text{ and } x \in P(s)$$

$$p^T b \leq c^T x \quad (\text{weak duality})$$

$$\Rightarrow p^T \cdot b \leq F(s) = \min_{x \in P(s)} c^T x$$

$$\Rightarrow F(s^*) - \underbrace{p^T \cdot b^* + p^T b}_{p^T \cdot (b - b^*)} \leq F(s)$$

$\Rightarrow$ : Assume that  $p$  is subgradient of  $F$  at  $s^*$ :

$$(*) \quad F(\delta^*) + p^T \cdot (\delta - \delta^*) \leq F(\delta) \quad \forall \delta \in S$$

For some  $x \geq 0$  let  $\delta := A \cdot x$  such that  $x \in P(\delta)$

$$\Rightarrow F(\delta) \leq c^T x \quad (**)$$

$$\Rightarrow p^T \cdot A \cdot x = p^T \cdot \delta \stackrel{(*)}{\leq} F(\delta) - F(\delta^*) + p^T \cdot \delta^*$$

$$\stackrel{(**)}{\leq} c^T \cdot x - F(\delta^*) + p^T \cdot \delta^* \quad \forall x \geq 0$$

Since this linear inequality holds for all  $x \geq 0$ , it implies that

$$p^T \cdot A \leq c^T \quad (\text{dual feas.})$$

For  $x=0$ , we get  $0 \leq -F(\delta^*) + p^T \cdot \delta^*$

$\Rightarrow p$  is optimal (strong duality).  $\square$

