

Proof of Theorem 3.8:

" \Leftarrow " clear

" \Rightarrow ": $\text{opt} = -\infty \Rightarrow \exists x \in C : c^T x < 0$
 $\Rightarrow \exists x \in C : c^T x = -1$

$$\Rightarrow P = \{x \in \mathbb{R}^n \mid a_i^T x \geq 0, i=1, \dots, m, c^T x = -1\} \neq \emptyset$$

P is pointed since C is pointed (i.e. a_1, \dots, a_m span \mathbb{R}^n)

Let $d \in P$ extreme point $\Rightarrow \exists$ u linearly indep. constraints active at d ,

$\Rightarrow u-1$ linearly indep. constr. of the form $a_i^T x \geq 0$ active at d

\Rightarrow Since $d \neq 0$ ($c^T d = -1$), d is thus an extreme ray of C . \square

Proof of Theorem 3.3: $P = \{x \mid Ax \geq b\}$

" \Leftarrow " clear.

" \Rightarrow ": Consider infeasible dual LP:

$$\begin{aligned} \max & p^T b \\ \text{st.} & p^T A = c^T \\ & p \geq 0 \end{aligned}$$

Replace obj. funct. by $p^T \cdot 0$
 \rightarrow problem remains infeasible

Take the corresp. primal LP:

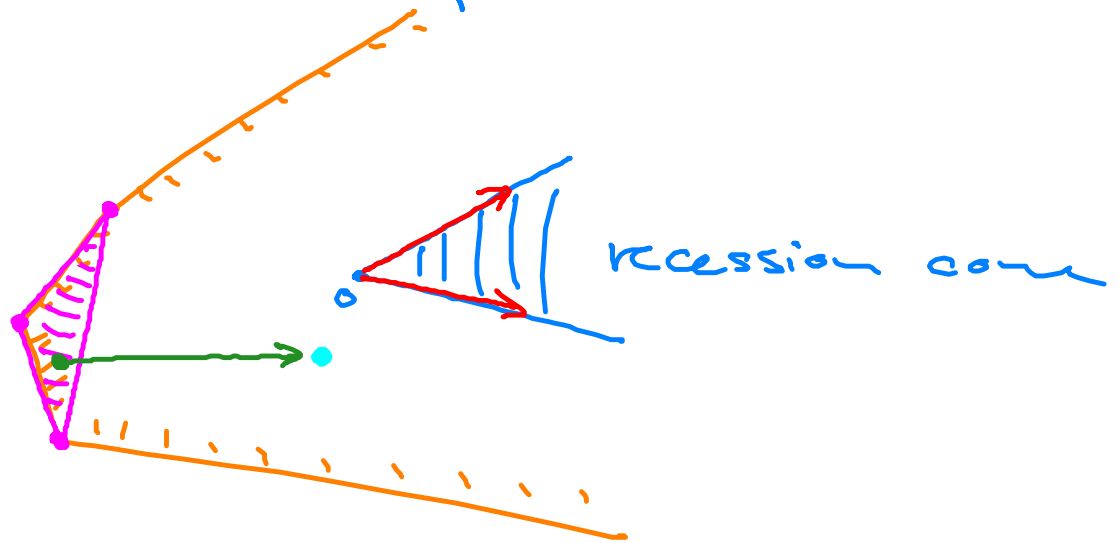
$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq 0 \end{aligned}$$

unbounded
(because feasible)

Due to Theorem 3.8 there is an extreme ray d of $\underbrace{\{x \mid Ax \geq 0\}}_{\text{recession cone of } P}$ with $c^T d < 0$

$\Rightarrow d$ is extreme ray of P . □

Theorem 3.10: Example



Proof of Th. 3.10:

$$Q := \text{r.h.s.}$$

" \supseteq " $P \supseteq Q$ clear by convexity of P and by definition of rays of P

" \subseteq " $P \subseteq Q$: Assume by contradiction that there is $z \in P \setminus Q$.

Consider the following LP:

$$\max \sum_{i=1}^k 0 \cdot \lambda_i + \sum_{j=1}^r 0 \cdot \theta_j$$

$$\text{s.t.} \quad \sum_{i=1}^k \lambda_i \cdot x^i + \sum_{j=1}^r \theta_j \cdot w^j = z$$

$$\sum_{i=1}^k \lambda_i = 1$$

$$\lambda, \theta \geq 0$$

infeasible
by choice
of z !

Take the dual:

$$\min p^T \cdot z + q$$

$$\text{s.t.} \quad p^T \cdot x^i + q \geq 0 \quad \text{for } i=1, \dots, k$$

$$p^T \cdot w^j \geq 0 \quad \text{for } j=1, \dots, r$$

Notice that $p=0, q=0$ is feasible solution
 \Rightarrow dual LP is unbounded!

$\Rightarrow \exists$ solution (p, q) with $p^T \cdot z + q < 0$

$$\Rightarrow p^T \cdot z < p^T x^i \quad \forall i$$

$$p^T w^j \geq 0 \quad \forall j$$

For this fixed vector p , consider the following LP:

$$\min p^T \cdot x$$

$$\text{s.t.} \quad Ax \geq b$$

Notice that z is a feasible solution!

1. Case: The LP has finite optimal cost
 $\rightarrow \exists$ opt. extreme point x^i for some i
 But $p^T z < p^T x^i \quad \Downarrow$

2. Case: The LP is unbounded

Th. 9.9
 $\implies \exists$ extreme ray w^i with $p^T \cdot w^i < 0 \quad \Downarrow$

Proof of Theorem 9.14:

For some $z \in \mathbb{R}^k$, consider the LP:

$$\max \sum_{i=1}^k 0 \cdot x_i + \sum_{j=1}^r 0 \cdot \theta_j$$

$$\text{s.t.} \quad \sum_{i=1}^k \lambda_i \cdot x^i + \sum_{j=1}^r \theta_j \cdot w^j = z$$

$$\sum_{i=1}^k \lambda_i = 1$$

$$\lambda_i, \theta_j \geq 0$$

Notice that $z \in Q \iff$ LP feasible and has finite opt. sol.

Consider dual LP:

$$\min p^T \cdot z + q$$

$$\text{s.t.} \quad p^T \cdot x^i + q \geq 0 \quad i=1, \dots, k$$

$$p^T \cdot w^j \geq 0 \quad j=1, \dots, r$$

$z \in Q \iff$ dual LP has finite opt.

Convert dual LP to standard form:

(write $p = p^+ - p^-$, $q = q^+ - q^-$, introduce slack var. α_i and β_j)

$$\min (p^+ - p^-)^T \cdot z + q^+ - q^-$$

$$\text{s.t. } (p^+ - p^-)^T \cdot x^i + q^+ - q^- - \alpha_i = 0 \quad i=1, \dots, k$$

$$(p^+ - p^-)^T \cdot w^j - \beta_j = 0 \quad j=1, \dots, r$$

$$p^+, p^-, q^+, q^-, \alpha, \beta \geq 0$$

Since LPs in standard form have a pointed feasible region. Thus, by Th. 3.8, the LP has finite opt. sol. if and only if

$$(p^+ - p^-)^T \cdot z + q^+ - q^- \geq 0 \quad (*)$$

for all extreme rays (finitely many) $(p^+, p^-, q^+, q^-, \alpha, \beta)$

Conclusion: $z \in Q \iff z$ fulfills the finitely many inequalities (*).

$\implies Q$ polyhedron. □