Proof of Th. 3.1:

"$\Rightarrow$": Let $x \in \mathbb{R}^n$: $Ax \leq b$

Suppose there is $y \geq 0$ with $y^T A = 0$
$y^T \cdot b < 0$

$$\Rightarrow \quad 0 > y^T b \geq y^T \cdot (A \cdot x) = (y^T \cdot A) \cdot x = 0 \ \lightning$$

"$\Leftarrow$": Suppose that $A \cdot x \leq b$ has no solution x.

Prove by induction on # columns of A $(=n)$ that there is $y \geq 0$ with $y^T A = 0^T$ and $y^T \cdot b < 0$.

$n=1$: clear.

$n-1 \to n$: $Ax \leq b$ has no solution

$\Rightarrow$ $A' x' \leq b'$ has no solution

$\overset{\text{Ind.}}{\Rightarrow} \exists y' \geq 0$: $y'^T A' = 0^T$ and $y'^T \cdot b' < 0$

Notice that $A' = U \cdot A$ for some matrix $U \geq 0$

$b' = U \cdot b$

Set $y^T := y'^T \cdot U \geq 0$

$$\Rightarrow y^T \cdot A = (y'^T \cdot U) \cdot A = y'^T \cdot (U \cdot A)$$

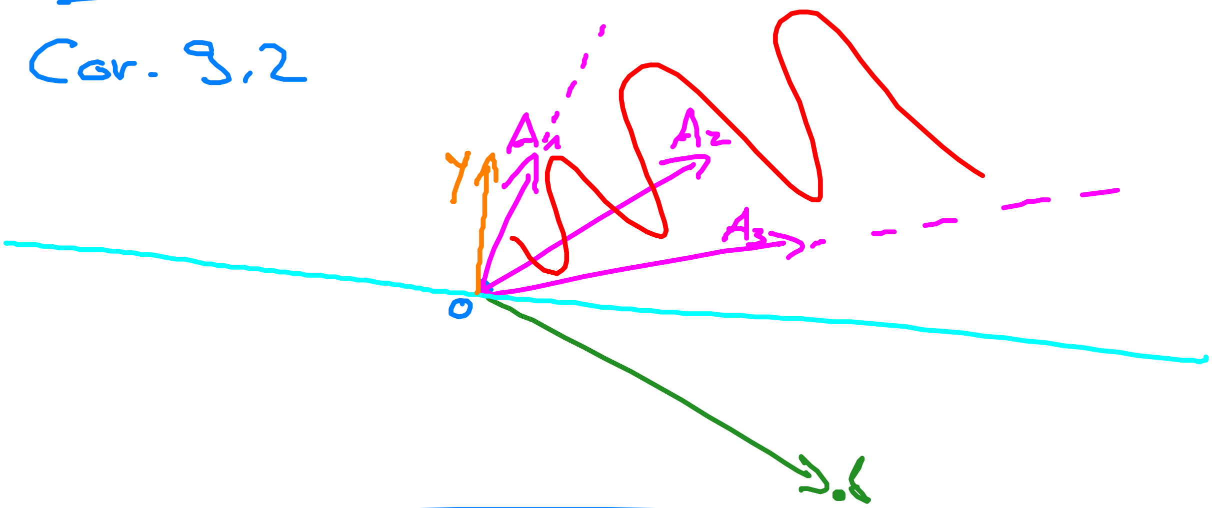
$$= y'^T \cdot A' = 0^T$$

$$y^T \cdot b = (y'^T \cdot U) \cdot b = y'^T \cdot (U \cdot b)$$

$$= y'^T \cdot b' < 0.$$

$\square$

Cor. 3.2



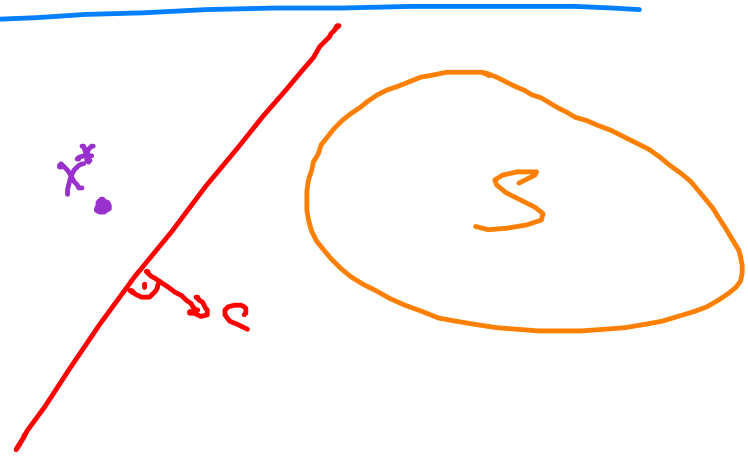
Proof of Cor. 3.2 (Farkas' Lemma) using LP duality:

Assume that there is no $x \geq 0 : Ax = b$:

Consider infeasible LP:

$$\begin{array}{ll} \max\ 0^T \cdot x & \qquad \min\ y^T \cdot b \\ \text{s.t. } A \cdot x = b & \qquad \text{s.t. } y^T \cdot A \geq 0^T \\ \qquad x \geq 0 & \end{array}$$

Dual LP is feasible (e.g. $y = 0$) and hence unbounded $\Rightarrow \exists\, y : y^T b < 0,\ y^T \cdot A \geq 0^T$ $\square$



Theorem 3.3 does not hold for nonconvex S:

Proof of Theorem 3.3:

Let $w \in S$ arbitrary and

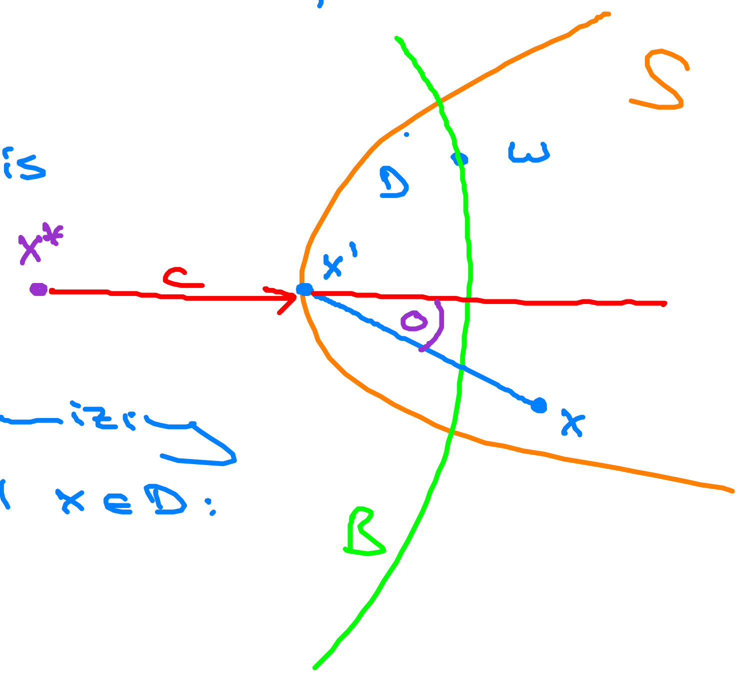
$$B = \{x \mid \|x - x^*\|_2 \leq \|w - x^*\|_2\}$$

Set $D := S \cap B$

Notice that D is bounded, closed and non-empty.

$\Rightarrow \exists\, x' \in D$ minimizing $\|x - x^*\|_2$ over all $x \in D$:

($f(x) := \|x - x^*\|_2$ continuous)



$$\|x' - x^*\|_2 \leq \|x - x^*\|_2 \quad \forall x \in D$$

even $\forall x \in S$

Set $c := x' - x^*$

Let $x \in S$ and $\lambda \in (0,1] \Rightarrow x' + \lambda \cdot (x - x') \in S$

$$\underline{\|x' - x^*\|_2^2} \leq \|x' + \lambda \cdot (x - x') - x^*\|_2^2$$

$$= \underline{\|x' - x^*\|^2} + 2\lambda \cdot (x' - x^*)^T \cdot (x - x') + \lambda^2 \|x - x'\|^2$$

$$\Rightarrow 2\cancel{\lambda} \cdot (x' - x^*)^T \cdot (x - x') + \lambda^{\cancel{2}} \cdot \|x - x'\|^2 \geq 0$$

Let λ go to 0: $\underbrace{(x' - x^*)^T}_{c^T} \cdot (x - x') \geq 0$

$$\Rightarrow \underbrace{(x' - x^*)}_{c}{}^T \cdot x \geq \underbrace{(x' - x^*)}_{c}{}^T \cdot x'$$

$$= c^T \cdot x^* + c^T \cdot \underbrace{(x' - x^*)}_{c}$$

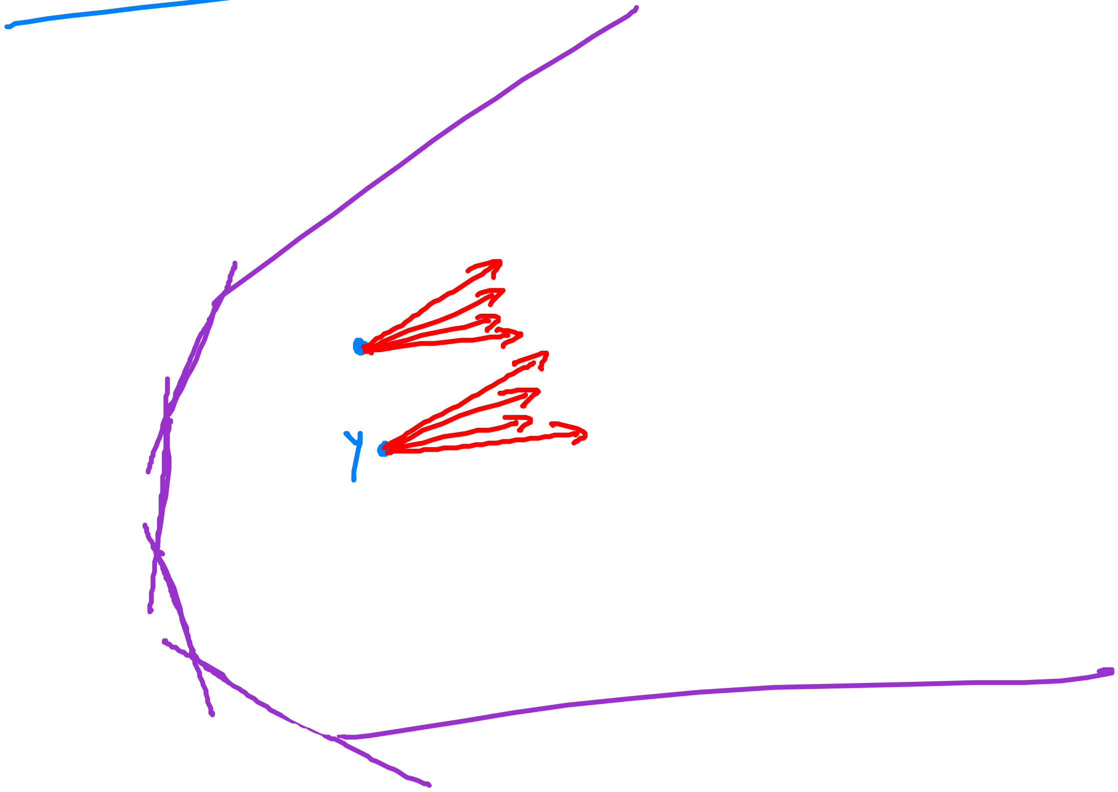
$$> c^T \cdot x^*$$

$\square$

From Theorem 3.3 one can easily obtain Farkas' Lemma (see exercises).

Let $x \neq 0$ with $Ax \geq 0$

Write $x = \frac{1}{2} \cdot \underbrace{0}_{\in P} + \frac{1}{2} \cdot \underbrace{(2x)}_{\in P}$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