Proof of Lemma 16.13:

For $0 \le r \le S$, let $X_r := (X \setminus \{e_1, \dots, e_r\}) \cup \{f_1, \dots, f_r\}$

Prove $X_r \in \mathcal{F}$ by induction on r:

$\underline{r = 0}$: $X_0 = X \in \mathcal{F}$

$\underline{r - 1 \longrightarrow r}$: $X_{r-1} \in \mathcal{F}$. If $X_{r-1} \cup \{f_r\} \in \mathcal{F} \Longrightarrow X_r \in \mathcal{F}$
Otherwise, $C(X, f_r) \subseteq (X \setminus \{e_1, \dots, e_{r-1}\}) \cup \{f_r\}$ by (ii), and thus $C(X_{r-1}, f_r) = C(X, f_r)$
$\Longrightarrow e_r \in C(X_{r-1}, f_r)$ by (i)
$\Longrightarrow X_r = (X_{r-1} \setminus \{e_r\}) \cup \{f_r\} \in \mathcal{F}$. q.e.d.

Proof of Lemma 16.14:

Show that $X \cup \{f_0\}$, $e_1, \dots, e_s$, and $f_1, \dots, f_s$ satisfy the requirements of Lemma 16.13 with respect to $\mathcal{F}_1$.

$X \cup \{f_0\} \in \mathcal{F}_1$ since $f_0 \in S_X$
requirement (i) holds since $(e_k, f_k) \in A_X^{(1)}$ for all k
requirement (ii) holds since otherwise there is a shortcut (e_j, f_k), $j < k$

By symmetry, X' is also in $\mathcal{F}_2$ q.e.d.

Proof of Lemma 16.15:

Proposition: For $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ and $Q \subseteq E$:

$$|F| \le r_2(Q) + r_1(E \setminus Q) \ .$$

Proof: $|F \cap Q| \le r_2(Q)$ and $|F \setminus Q| \le r_1(E \setminus Q)$. q.e.d. (proposition)

$\underline{\Longrightarrow}$: Follows from Lemma 16.14.

$\underline{\Longleftarrow}$: Let $R := \{\text{nodes reachable from } S_x \text{ in } D_X\}$
$\Longrightarrow R \cap T_X = \emptyset$

Claim: $r_2(R) = |X \cap R|$

Proof: Otherwise: $\exists\ f \in R \setminus X$: $(X \cap R) \cup \{f\} \in \mathcal{F}_2$
Since $X \cup \{f\} \notin \mathcal{F}_2$ ($f \notin T_X$), the circuit $C_2(X, f)$ must contain $e \in X \setminus R$
$\Longrightarrow (f, e) \in A_X^{(2)}$ (contradiction to $f \in R$, $e \notin R$) q.e.d. (claim)

Claim: $r_1(E \setminus R) = |X \setminus R|$

Proof: as above q.e.d. (claim)

$\Longrightarrow |X| = |X \cap R| + |X \setminus R| = r_2(R) + r_1(E \setminus R)$ q.e.d.