

# Proof of Theorem 16.4:

(i)  $\Rightarrow$  (ii) clear.

(ii)  $\Rightarrow$  (iii) Assume that  $B_1, B_2 \subseteq X \subseteq E$   
are bases of different card., i.e.

$$|B_1| > |B_2|.$$

Take  $Y \in B_1$  with  $|Y| = |B_2| + 1$

$\stackrel{(ii)}{\Rightarrow} \exists e \in Y \setminus B_2 : \underbrace{B_2 \cup \{e\}}_{\subseteq X} \in \mathfrak{F} \quad \begin{matrix} \swarrow \\ \downarrow \end{matrix} \begin{matrix} B_2 \text{ basis} \\ \text{of } X. \end{matrix}$

(iii)  $\Rightarrow$  (i): Let  $X, Y \in \mathfrak{F}$ ,  $|X| > |Y|$

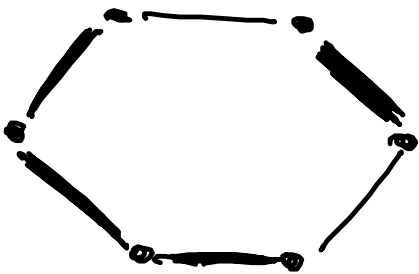
Since all bases of  $X \cup Y$  have the same cardinality,  $Y$  cannot be a basis

$\Rightarrow \exists e \in X \setminus Y : Y \cup \{e\} \in \mathfrak{F}$ . □

## Example:

$E =$  set of edges

$\mathfrak{F} = \{\text{matchings}\}$



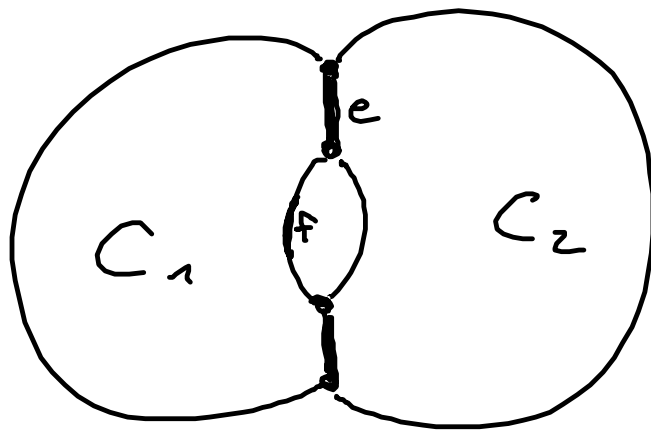
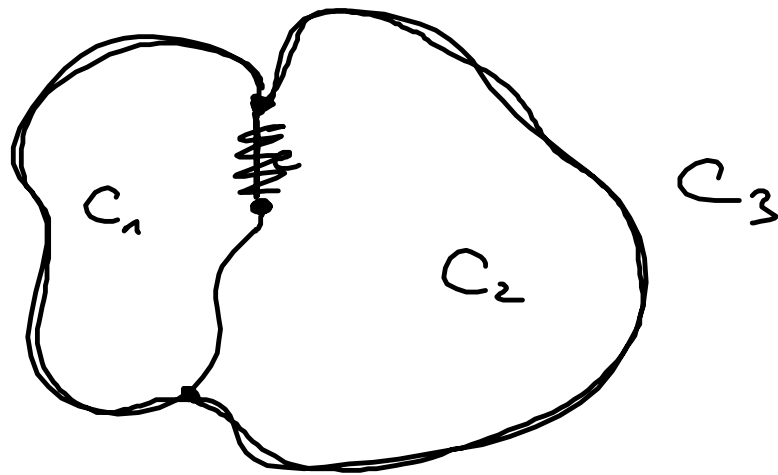
$$r(E) = 3$$

$$g(E) = 2$$

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rank quotient:  $\frac{1}{2}$



Proof of Theorem 16.11:

Let  $G_n \subseteq E$  be the greedy solution  
and  $O_n \subseteq E$  an opt. solution.

Let  $E_j := \{1, \dots, j\}$ ,  $G_j := G_n \cap E_j$

$O_j := O_n \cap E_j$

for  $j = 0, \dots, n$

$$\begin{aligned}
c(G_n) &= \sum_{j=1}^n (|G_j| - |G_{j-1}|) \cdot c(e_j) \\
&= \sum_{j=1}^n |G_j| \cdot d_j \quad \text{with } d_j := c(e_j) - c(e_{j+1}) \\
&\qquad\qquad\qquad \geq 0 \\
&\qquad\qquad\qquad d_n := c(e_n) \\
&\geq \sum_{j=1}^n g(E_j) \cdot d_j \\
&\geq q(E, \tilde{\mathcal{F}}) \cdot \sum_{j=1}^n r(E_j) \cdot d_j \\
&\geq q(E, \tilde{\mathcal{F}}) \cdot \sum_{j=1}^n |\sigma_j| \cdot d_j \\
&= q(E, \tilde{\mathcal{F}}) \cdot \sum_{j=1}^n (|\sigma_j| - |\sigma_{j-1}|) \cdot c(e_j) \\
&= q(E, \tilde{\mathcal{F}}) \cdot c(\sigma_n)
\end{aligned}$$

To prove the tightness of the lower bound, let  $F \subseteq E$  with bases  $B_1, B_2$  of  $F$  such that  $\frac{|B_1|}{|B_2|} = q(E, \tilde{\mathcal{F}})$ .

Let  $c(e) := \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{otherwise} \end{cases}$

Sort  $E$  such that  $c(e_1) \geq c(e_2) \geq \dots \geq c(e_n)$  and  $B_1 = \{e_1, \dots, e_{|B_1|}\}$ . Then,

$$G(E, \tilde{\mathcal{F}}, c) = |B_1| \quad \text{and} \quad \text{OPT}(E, \tilde{\mathcal{F}}, c) = |B_2|$$

□

# Proof of Theorem 16.12:

" $\Leftarrow$ " clear.

" $\Rightarrow$ ": Show that all vertices of the given polytope are integral.

Let  $c \in \mathbb{R}^E$  and show that

$$\max \{ c^T x \mid x \geq 0, \sum_{e \in X} x_e \leq r(X) \forall X \subseteq E \}$$

has integral optimum solution  $x$ .

w.l.o.g.  $c \geq 0$

$$c(G_n) = \sum_{j=1}^n r(E_j) \cdot d_j \quad (\text{see above})$$

$$\geq \sum_{j=1}^n \left( \sum_{e \in E_j} x_e \right) \cdot d_j$$

$$= \sum_{e \in E} c_e \cdot x_e = c^T x$$

□