S is convex:

$b^1, b^2 \in S$, $0 \le \lambda \le 1$

Prove that $\lambda \cdot b^1 + (1-\lambda) \cdot b^2 \in S$

Write $b^1 = A \cdot x^1$, $x^1 \geq 0$

$b^2 = A \cdot x^2$, $x^2 \geq 0$

Set $\bar{x} := \lambda \cdot x^1 + (1-\lambda) \cdot x^2 \geq 0$

$$S \ni A \cdot \bar{x} = \lambda \cdot A \cdot x^1 + (1-\lambda) \cdot A \cdot x^2$$
$$= \lambda \cdot b^1 + (1-\lambda) \cdot b^2 \qquad \square$$

Proof of Theorem 10.1:

Second part (linearity and gradient) clear.

First part: Consider the feasible dual program:

$$\max \ p^T \cdot b \qquad \text{s.t.} \ p^T A \le c^T$$

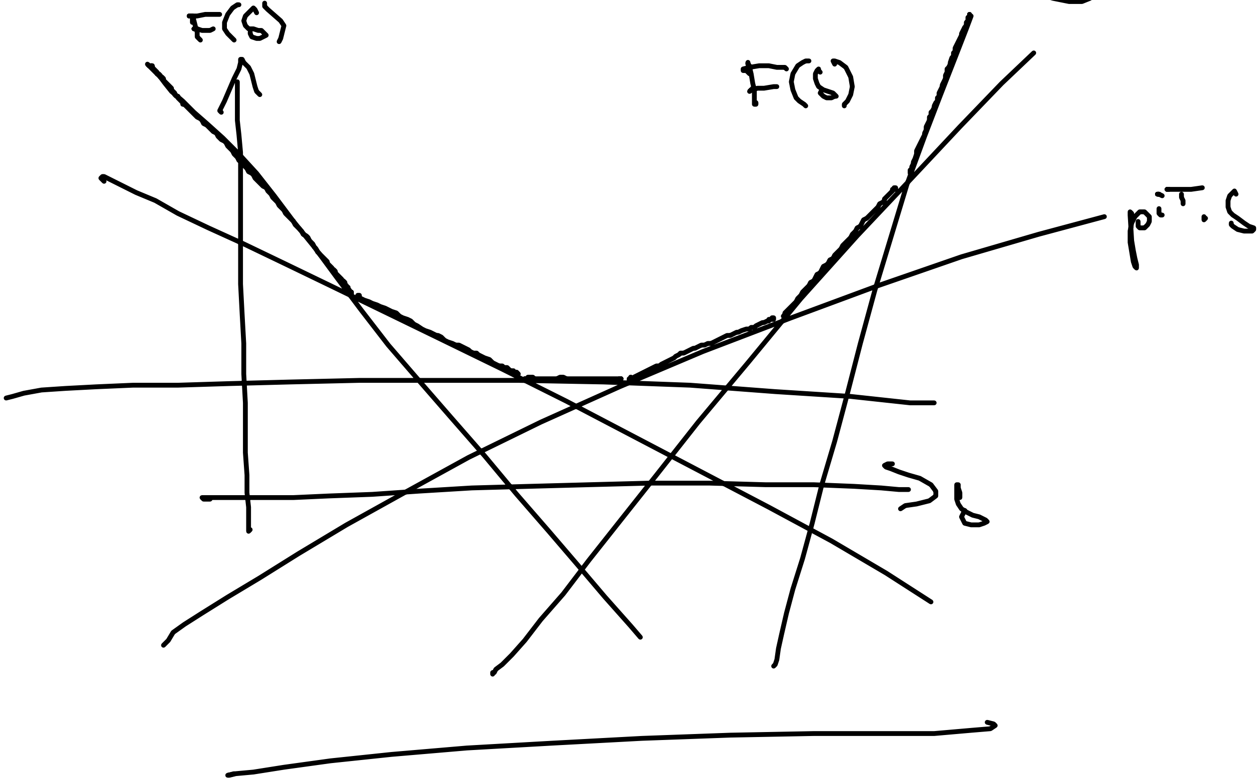
$\operatorname{rank}(A) = m$
$\Rightarrow$ extreme points do exist

Let $p^1, \ldots, p^N$ be the extreme points of $\{p \mid p^T A \le c^T\}$. By strong duality:

$$F(b) = \max_{i=1,\ldots,N} p^{i^T} \cdot b \quad \text{for } b \in S$$

Notice that a linear function is convex

and the maximum of convex functions is a convex function again. $\square$



<u>Proof of Theorem 10.3:</u>

<u>"⇒":</u> Let p be an opt. dual solution

$\Rightarrow p^T \cdot b^* = F(b^*)$. For $b \in S$ and $x \in P(b)$

$$p^T b \le c^T x \quad \text{(weak duality)}$$

$$\Rightarrow p^T \cdot b \le F(b) = \min_{x \in P(b)} c^T x$$

$$\Rightarrow F(b^*) \underbrace{- p^T \cdot b^* + p^T b}_{p^T \cdot (b - b^*)} \le F(b)$$

<u>"⇐":</u> Assume that p is subgradient of F at b^*:

$$(*)\quad F(\delta^*) + p^T \cdot (\delta - \delta^*) \le F(\delta) \quad \forall \delta \in S$$

For some $x \ge 0$ let $\delta := A \cdot x$ such that $x \in P(\delta)$

$$\Rightarrow F(\delta) \le c^T x \quad (**)$$

$$\Rightarrow p^T \cdot A \cdot x = p^T \cdot \delta \overset{(*)}{\le} F(\delta) - F(\delta^*) + p^T \cdot \delta^*$$

$$\overset{(**)}{\le} c^T \cdot x - F(\delta^*) + p^T \cdot \delta^* \quad \forall x \ge 0$$

Since this linear inequality holds for all $x \ge 0$, it implies that

$$p^T \cdot A \le c^T \quad \text{(dual feas.)}$$

For $x = 0$, we get $0 \le -F(\delta^*) + p^T \cdot \delta^*$

$\Rightarrow p$ is optimal (strong duality). $\square$