Proof of Theorem 9.8:

"$\Longleftarrow$" clear

"$\Longrightarrow$": $opt = -\infty \Rightarrow \exists x \in C : c^T x < 0$
$\Rightarrow \exists x \in C : c^T x = -1$

$$\Rightarrow P = \{x \in \mathbb{R}^n \mid a_i^T \cdot x \geq 0, \; i = 1, \ldots, m, \; c^T x = -1\} \neq \emptyset$$

P is pointed since C is pointed (i.e. $a_1, \ldots, a_m$ span $\mathbb{R}^n$)

Let $d \in P$ extreme point $\Rightarrow \exists$ n linearly indep. constraints active at d,
$\Rightarrow n-1$ linearly indep. constr. of the form $a_i^T x \geq 0$ active at d
$\Rightarrow$ Since $d \neq 0$ $(c^T d = -1)$, d is thus an extreme ray of C. $\square$

Proof of Theorem 9.9: $P = \{x \mid Ax \geq b\}$

"$\Longleftarrow$" clear.

"$\Longrightarrow$": Consider infeasible dual LP:

$$\begin{aligned} &\max \; p^T \cdot b \\ &\text{s.t. } p^T \cdot A = c^T \\ &\qquad p \geq 0 \end{aligned}$$

Replace obj. funct. by $p^T \cdot 0$
$\rightarrow$ problem remains infeasible

Take the corresp. primal LP:

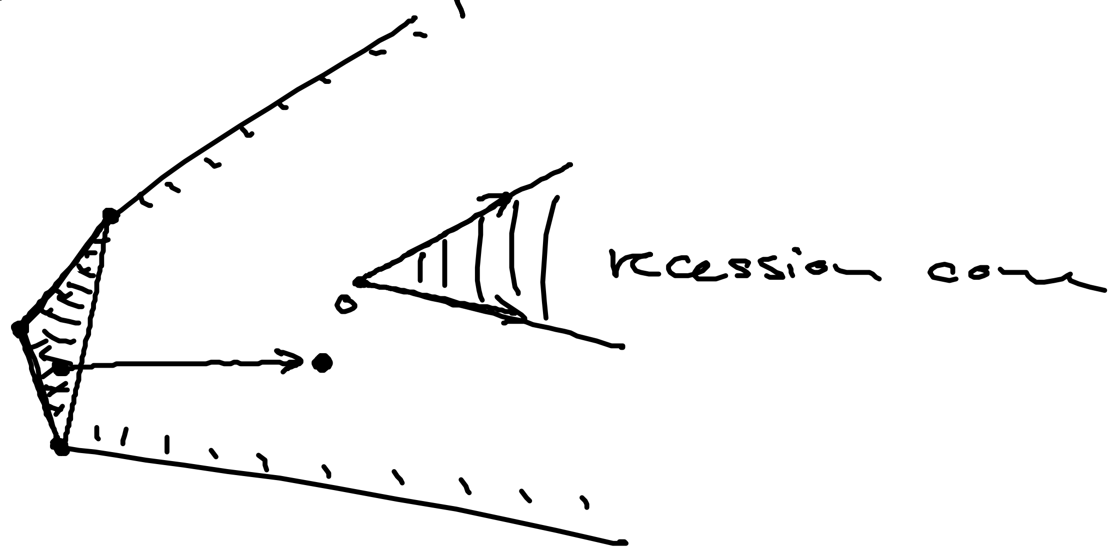
$$\min c^T x$$
$$\text{s.t. } Ax \geq 0$$

unbounded (because feasible)

Due to Theorem 3.8 there is an extreme ray d of $\underbrace{\{x \mid Ax \geq 0\}}_{\text{recession cone of } P}$ with $c^T d < 0$

$\Rightarrow d$ is extreme ray of P. $\square$

Theorem 3.10: Example



<u>Proof of Th. 3.10:</u>

$Q :=$ r.h.s.

„$\supseteq$" $P \supseteq Q$ clear by convexity of P and by definition of rays of P

„$\subseteq$" $P \subseteq Q$: Assume by contradiction that there is $z \in P \setminus Q$.

Consider the following LP:

$$\max \sum_{i=1}^{k} 0 \cdot \lambda_i + \sum_{j=1}^{r} 0 \cdot \theta_j$$

$$\text{s.t. } \sum_{i=1}^{k} \lambda_i \cdot x^i + \sum_{j=1}^{r} \theta_j \cdot w^j = z$$

$$\sum_{i=1}^{k} \lambda_i = 1$$

$$\lambda, \theta \geq 0$$

infeasible by choice of z!

Take the dual:

$$\min p^T \cdot z + q$$

$$\text{s.t. } p^T \cdot x^i + q \geq 0 \quad \text{for } i = 1, \ldots, k$$

$$p^T \cdot w^j \geq 0 \quad \text{for } j = 1, \ldots, r$$

Notice that $p = 0, q = 0$ is feasible solution

$\Rightarrow$ dual LP is unbounded!

$\Rightarrow \exists$ solution (p, q) with $p^T \cdot z + q < 0$

$$\Rightarrow p^T \cdot z < p^T x^i \quad \forall i$$

$$p^T w^j \geq 0 \quad \forall j$$

For this fixed vector p, consider the following LP:

$$\min p^T \cdot x$$

$$\text{s.t. } Ax \geq b$$

Notice that z is a feasible solution!

1. Case: The LP has finite optimal cost

$\Rightarrow \exists$ opt. extreme point x^i for some i

But $p^T z < p^T x^i$ $\lightning$

2. Case: The LP is unbounded

$\overset{\text{Th. 9.9}}{\Longrightarrow}$ $\exists$ extreme ray w^j with $p^T \cdot w^j < 0$ $\lightning$

$\square$

Proof of Theorem 9.14:

For some $z \in \mathbb{R}^n$, consider the LP:

$$\max \sum_{i=1}^{k} 0 \cdot \lambda_i + \sum_{j=1}^{r} 0 \cdot \theta_j$$

$$\text{s.t.} \sum_{i=1}^{k} \lambda_i \cdot x^i + \sum_{j=1}^{r} \theta_j \cdot w^j = z$$

$$\sum_{i=1}^{k} \lambda_i = 1$$

$$\lambda, \theta \geq 0$$

Notice that $z \in Q \Leftrightarrow$ LP feasible and has finite opt. sol.

Consider dual LP:

$$\min p^T \cdot z + q$$

$$\text{s.t. } p^T \cdot x^i + q \geq 0 \quad i = 1, \dots, k$$

$$p^T \cdot w^j \geq 0 \quad j = 1, \dots, r$$

$z \in Q \Leftrightarrow$ dual LP has finite opt.

Convert dual LP to standard form:

(write $p = p^+ - p^-$, $q = q^+ - q^-$, introduce slack var. α_i and β_j)

$$\min \ (p^+ - p^-)^T \cdot z + q^+ - q^-$$

$$\text{s.t.} \ (p^+ - p^-)^T \cdot x^i + q^+ - q^- - \alpha_i = 0 \qquad i = 1, \dots, k$$

$$(p^+ - p^-)^T \cdot w^j - \beta_j = 0 \qquad j = 1, \dots, r$$

$$p^+, p^-, q^+, q^-, \alpha, \beta \geq 0$$

Since LPs in standard form have a pointed feasible region. Thus, by Th. 3.8, the LP has finite opt. sol. if and only if

$$(p^+ - p^-)^T \cdot z + q^+ - q^- \geq 0 \qquad (*)$$

for all finitely many extreme rays $(p^+, p^-, q^+, q^-, \alpha, \beta)$

<u>Conclusion</u>: $z \in Q \iff z$ fulfills the finitely many inequalities $(*)$.

$\implies Q$ polyhedron. $\square$