Technische Universität Berlin
Institut für Mathematik
Dr. Britta Peis
Prof. Dr. Martin Skutella
Ágnes Cseh

Discrete Optimization (ADM II)
SoSe 2012

Eleventh and Last Set of Exercises

(due date: July 5, 12:00, in room MA 501)

Exercise 67 (4 points)**.** Let E be a finite set, $c : E \to \mathbb{R} \setminus \{0\}$ and $\mathcal{F} \subseteq 2^E$ such that $(E, \mathcal{F})$ is a matroid. Moreover, let $X_1, X_2 \in \mathcal{F}$ be two optimal solutions to the corresponding maximization problem, i. e.,

$$c(X_1) = c(X_2) = \max\{c(X) \mid X \in \mathcal{F}\} \ ,$$

and $e \in X_1$. Prove that there exists an $e' \in X_2$ with $c(e') = c(e)$. In particular, if $c(e_1) \neq c(e_2)$ for all $e_1 \neq e_2$, then there exists a unique optimal solution.

Exercise 68 (4 points)**.** Let $G = (V, E)$ be an undirected graph and

$$\mathcal{F} := \{F \subseteq V \mid \text{there is a matching } M \text{ of } G \text{ covering all nodes in } F\} \ .$$

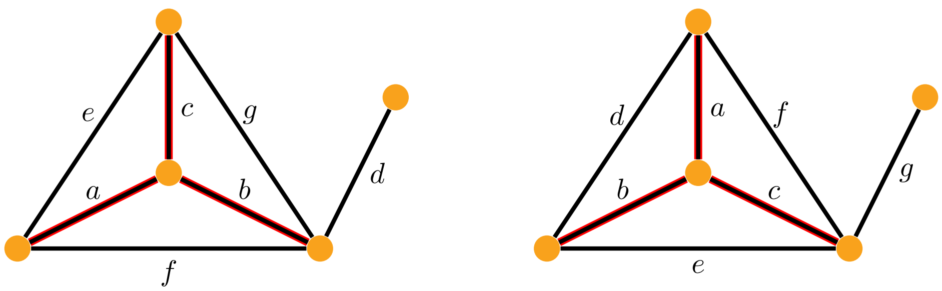
Prove that $(V, \mathcal{F})$ is a matroid.

Exercise 69 (4 points)**.** Prove Theorem 16.8:
Let E be a finite set and $\mathcal{B} \subseteq 2^E$. Then, $\mathcal{B}$ is the set of bases of some matroid $(E, \mathcal{F})$ if and only if

i) $\mathcal{B} \neq \emptyset$;

ii) $B_1, B_2 \in \mathcal{B}$, $e \in B_1 \setminus B_2 \implies \exists f \in B_2 \setminus B_1$ with $(B_1 \setminus \{e\}) \cup \{f\} \in \mathcal{B}$.

Exercise 70 (4 points)**.** Consider the matroid intersection problem on the graphic matroids defined by the following graphs:



For $X := \{a, b, c\}$, construct the digraph D_X. Find an S_X-T_X-path for which the corresponding augmentation does *not* yield a common independent set. Find a shortest S_X-T_X-path and construct an optimum solution to the matroid intersection problem.

Exercise 71 (4 points)**.** Prove that the rank function of a matroid is submodular.

Exercise 72 (tutorial session)**.** Let E be a finite set, $c : E \to \mathbb{R}$ and $\mathcal{F} \subseteq 2^E$ such that $(E, \mathcal{F})$ is a matroid. Prove that the greedy algorithm yields a basis B of E maximizing $\min\{c(e) \mid e \in B\}$.

Exercise 73 (tutorial session)**.** Let $G = (V, E)$ be a bipartite graph and

$$\mathcal{F} := \{M \subseteq E \mid M \text{ is a matching in } G\} \ .$$

Prove that the independence system $(E, \mathcal{F})$ is the intersection of two matroids.

Exercise 74 (tutorial session)**.** Let E be a finite set and $f : 2^E \to \mathbb{R}$. Prove that the following three statements are equivalent:

i) $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for all $A, B \subseteq E$;

ii) $f(S \cup \{j\}) - f(S) \geq f(T \cup \{j\}) - f(T)$ for all $j \in E$ and $S \subseteq T \subseteq E \setminus \{j\}$;

iii) $f(S \cup \{j\}) - f(S) \geq f(S \cup \{j, k\}) - f(S \cup \{k\})$ for all $j \neq k \in E$ and $S \subseteq E \setminus \{j, k\}$.

A function f with these properties is called *submodular*.