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Discrete Optimization (ADM II)
SoSe 2012

Tenth Set of Exercises

(due date: June 28, **before** the exercise session)

Exercise 60 (2+1 points)**.** Given a connected, undirected graph $G = (V, E)$ and edge-capacities $u_e > 0$ $(e \in E)$, a *legal ordering* of G is an ordering $V = \{v_1, v_2, \dots, v_n\}$ such that, where $V_i := \{v_1, \dots, v_i\}$,

$$u(\delta(V_{i-1}) \cap \delta(v_i)) \geq u(\delta(V_{i-1}) \cap \delta(v_j)) \quad \forall 2 \leq i < j \leq n.$$

a) Describe an algorithm that finds a legal ordering in G. (Runtime?)

b) Find a legal ordering of graph G as illustrated in Exercise 59, starting with $v_1 = r$.

Exercise 61 (2+3+3 points)**.** Given a connected, undirected graph $G = (V, E)$ and edge-capacities $u_e > 0$ $(e \in E)$,

a) Show that for any triple $u, v, w \in V$ holds $\mu(G; u, v) \geq \min\{\mu(G; w, u), \mu(G; w, v)\}$.

b) Show that $u(\delta(v_n)) = \mu(G; v_n, v_{n-1})$ holds for any legal ordering $\{v_1, v_2, \dots, v_n\}$ of the nodes in G.

c) Describe an algorithm to find a global min cut in time $\mathcal{O}(|V|^3)$ that does not need any maximum flow computation.

Exercise 62 (4 points)**.** Let $G = (V, E)$ be an undirected graph with edge costs $c \in \mathbb{R}^E$ and $v_1 \in V$. Show that Dantzig, Fulkerson, and Johnson's relaxation of the TSP (see Lecture, LP 14.1) is equivalent to

$$\begin{array}{llcll} \min & \sum_{e\in E} c_e x_e & & & \\ \text{s.t.} & x(\delta(v)) & = & 2 & \forall v \in V \\ & x(\gamma(S)) & \leq & |V|-1 & \forall S \subseteq V, v_1 \notin S \\ & 0 \leq x_e & & \leq 1 & \forall e \in E \end{array} \tag{1}$$

Exercise 63 (3+2 points)**.** Let $G = (V, E)$ be an undirected graph with edge costs $c \in \mathbb{R}^E$.

a) Show how to solve the linear program

$$\begin{array}{llcll} \min & \sum_{e\in E} c_e x_e & & & \\ \text{s.t.} & x(\delta(v)) & = & 2 & \forall v \in V \\ & 0 \leq x_e & & \leq 1 & \forall e \in E \end{array} \tag{2}$$

as a minimum-cost flow problem.

b) Use your construction to prove that there exists an optimal solution for which all variables have value 0, $\frac{1}{2}$, or 1.

Exercise 64 (tutorial session)**.** A variant on the TSP permits a node to be visited more than once if it results in a better solution. Show that if the cost function satisfies the triangle-inequality, then there is always an optimum solution which visits each node exactly once.

Exercise 65 (tutorial session)**.** Use the following example to show that the approximation factor $\frac{3}{2}$ of Christofides' algorithm cannot be decreased:
Let $V = \{v_1, \ldots, v_n\}$ and define $c_{v_i v_j} := \lfloor \frac{|i-j|+1}{2} \rfloor$ for $i \neq j$

Exercise 66 (tutorial session)**.** Given a connected, undirected graph $G = (V, E)$ and edge-capacities $u_e > 0$ $(e \in E)$, the *(global) minimum cut problem* asks for an edge-set $A \subseteq E$ such that $\emptyset \subset S \subset V$ and $u(A)$ is minimum.

a) Show that the global min cut problem in G can be solved by solving $|V| - 1$ maximum flow problems. What is the runtime of this algorithm?

b) Let $\mu(G)$ denote the capacity of a minimum global cut in G. Moreover, given any pair $u, v \in V$, let $\mu(G; v, w)$ denote the capacity of a minimum (u, v)-cut in G, and G_{uv} be the graph obtained by identifying the nodes u and w to a single nodes (see Figure).
Show that $\mu(G) = \min\{\mu(G_{uv}), \mu(G; u, v)\}$.

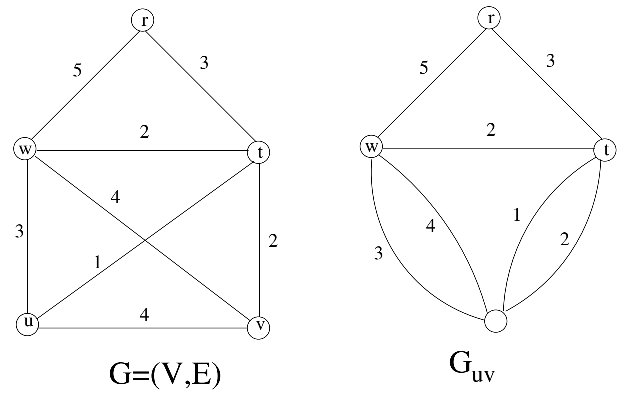

Figure 1: Node-identification.

c) Describe a faster algorithm to find a global min cut than the one proposed in a).