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Discrete Optimization (ADM II)
SoSe 2012

Sixth Set of Exercises

(due date: May 31, **before** the exercise session)

Exercise 32 (2+2+2 points)**.** Given a graph $G = (V, E)$ with edge weights $c \in \mathbb{R}^E$ and a perfect matching M of G, we form a digraph $D = (V, A)$ with arc weights $c' \in \mathbb{R}^A$ as follows. Let

$$A := \big\{(v, w) \mid \exists u \in V : \{v, u\} \in E, \{u, w\} \in M\big\} \qquad \text{and} \qquad c'_{(v,w)} := c_{\{v,u\}} - c_{\{u,w\}} \;.$$

Notice that the arc weights are well defined since node u justifying arc (v, w) in the definition is unique. Prove the following statements.

a) If M is not a minimum-weight perfect matching of G, then D has a negative-weight dicircuit.

b) If D has a dicircuit of negative weight and G is bipartite, then G has a perfect matching of smaller weight than M.

c) Statement b) is not true for general graphs.

Exercise 33 (2+3 points)**.** An *edge cover* of a graph $G = (V, E)$ (having no isolated nodes) is a subset of edges $D \subseteq E$ such that every node is incident to at least one edge in D.

a) Show how a minimum cardinality edge cover can be determined from a maximum matching. Prove that the minimum cardinality of an edge cover is $|V| - \nu(G)$.

b) Show that one can compute a minimum-weight edge cover by computing a maximum-weight matching.

Hint for b): Consider edge weights $c'_{\{u,v\}} := w_u + w_v - c_{\{u,v\}}$, where $w_v := \min_{e \in \delta(v)} c_e$.

Exercise 34 (5 points)**.** Consider a graph $G = (V, E)$ with edge weights $c \in \mathbb{R}^E_{\geq 0}$ and two distinguished nodes $s, t \in V$. We want to find a minimum-weight simple s-t-path in G having an odd number of edges. Make a copy $G' = (V', E')$ of G by setting $V' := \{v' \mid v \in V\}$ and $E' := \big\{e' = \{u', v'\} \mid e = \{u, v\} \in E\big\}$. Construct a new graph $\hat{G} = (\hat{V}, \hat{E})$ with

$$\hat{V} := V \cup V' \setminus \{s', t'\} \qquad \text{and} \qquad \hat{E} := \big\{\{v, v'\} \mid v \in V \setminus \{s, t\}\big\} \cup E \cup \big(E' \setminus \big(\delta_{G'}(s) \cup \delta_{G'}(t)\big)\big)$$

and edge weights $\hat{c} \in \mathbb{R}^{\hat{E}}_{\geq 0}$ given by $\hat{c}_e := \hat{c}_{e'} := c_e$, for $e \in E$, and $\hat{c}_{\{v,v'\}} := 0$ for $v \in V \setminus \{s, t\}$. Show that this shortest odd path problem in G can be solved by computing a minimum-weight perfect matching in $\hat{G}$. How can we find a minimum-weight simple s-t-path in G having an even number of edges?

Exercise 35 (4 points)**.** Let $G = (V, E)$ be a graph and suppose that G is connected and every node of G has even degree. Let P be an edge-simple closed path in G, and suppose that $E(P) \subsetneq E$. Show that there exists an edge-simple closed path P' in G with $E(P') \cap E(P) = \emptyset$ such that P' and P have at least one node in common. Use this idea to prove that a connected graph has an Euler tour if and only if every node of G has even degree. Moreover, give an algorithm that constructs an Euler tour of such a graph G.

Exercise 36 (tutorial session)**.** Suppose we are given a set V of points in the plane $\mathbb{R}^2$, and we define the distance between points $u, v \in \mathbb{R}^2$ to be $\|u - v\|_\infty$. If we wish to obtain a lower bound for the solution of a minimum-weight perfect matching problem by packing control zones and moats, what shape should these now be?

Exercise 37 (tutorial session)**.** Prove that if every node of a given graph $G = (V, E)$ has even degree, then E is the union of edge-disjoint circuits.

Exercise 38 (tutorial session)**.** A *Eulerian path* in graph $G = (V, E)$ is an edge-simple path that visits every edge exactly once. Give a necessary and sufficient condition for the existence of a Eulerian path that is based on node degrees.