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Fifth Set of Exercises

(due date: May 24, **before** the exercise session)

Notice that this is the last set of exercises in the first half of the semester.

Exercise 23 (4 points)**.** Prove Birkhoff's Theorem by transforming the minimum-weight perfect matching problem in a bipartite graph to a minimum-cost flow problem.

Exercise 24 (4 points)**.** Let $n \in \mathbb{Z}_{>0}$. A matrix $A \in [0,1]^{n\times n}$ is *doubly stochastic* if the entries in each row and in each column sum up to 1. Show that each doubly stochastic matrix is a convex combination of permutation matrices.

Exercise 25 (4 points)**.** Let $G = (V,E)$ be an undirected graph and $c \in \mathbb{R}^E$. Consider the linear programming relaxation of the minimum-weight perfect matching problem:

$$\begin{aligned} \min \quad & \sum_{e\in E} c_e \cdot x_e \\ \text{s.t.} \quad & x\big(\delta(v)\big) = 1 \qquad \text{for all } v \in V \\ & x \geq 0 \end{aligned}$$

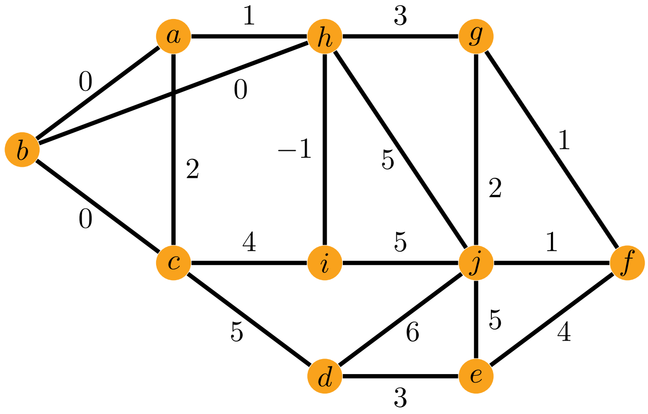
Show that there exists a feasible LP solution if and only if there exists a subset of edges and a set of odd circuits in G such that every node of G is a node of exactly one of the odd circuits or is incident to exactly one of the edges, but not both.

Exercise 26 (4 points)**.** Prove that the optimal value of the following problem is the minimum weight of a perfect matching of G.

$$\begin{aligned} \min \quad & \sum_{e\in E} c_e \cdot x_e \\ \text{s.t.} \quad & x\big(\delta(v)\big) = 1 && \text{for all } v \in V \\ & x\big(\gamma(S)\big) \leq (|S|-1)/2 && \text{for all } S \subseteq V,\ |S| \geq 3,\ |S| \text{ odd} \\ & x \geq 0 \end{aligned}$$

Exercise 27 (4 points)**.** Let M be a perfect matching of G and define the "cost" of an M-alternating circuit C of G to be $c\big(E(C)\setminus M\big) - c\big(E(C)\cap M\big)$. Prove that M is of minimum weight with respect to c if and only if there is no M-alternating circuit of negative cost.

Exercise 28 (tutorial session)**.** Find a minimum-weight perfect matching and an optimal solution of the dual problem for the following weighted graph:



Exercise 29 (tutorial session)**.** Let $G = (V, E)$ be an undirected graph, let $c \in \mathbb{R}^E$, and let k be a positive integer. Show how the problem of finding a minimum-weight matching having cardinality k can be reduced to a minimum-weight perfect matching problem.

Exercise 30 (tutorial session)**.** Show that there exists a non-bipartite graph with the property that, for every $c \in \mathbb{R}^E$, the optimal value of the linear program in Exercise 25 is equal to the minimum weight of a perfect matching of G.

Exercise 31 (tutorial session)**.** Let $G = (V, E)$ be an undirected graph. Prove that G has a perfect matching if and only if there is an $x \in \mathbb{R}^E$ with:

$$\begin{aligned} x\big(\delta(v)\big) &= 1 && \text{for all } v \in V \\ x\big(\delta(S)\big) &\geq 1 && \text{for all } S \subseteq V,\ |S| \text{ odd} \\ x &\geq 0 \end{aligned}$$